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Formal modeling and verification of fractional order linear systems

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ABSTRACT

This paper presents a formalization of a fractional order linear system in a higher-order logic (HOL) theorem proving system. Based on the formalization of the *Grünwald–Letnikov* (GL) definition, we formally specify and verify the linear and superposition properties of fractional order systems. The proof provides a rigor and solid underpinnings for verifying concrete fractional order linear control systems. Our implementation in HOL demonstrates the effectiveness of our approach in practical applications.

1. Introduction

The research on the theory of fractional calculus was initiated nearly as the same time as that of the integer order calculus. However, the research progress of fractional calculus had lagged that of integer order calculus, due to the lack of practical applications, until Mandelbort's work in 1983 [1]. It is important to know the existence of fractal-dimension numbers in nature [1]. Fractional calculus is a direct expansion of the traditional integer order calculus. It allows the order of the derivative of a function to be a fractional number instead of an integer [2]. In recent years, fractional calculus has proved to be of great value in developing fractional order PID controllers [3–5], signal processing [6], image processing [7], earthquake analysis [8], viscoelastic systems, etc. [9]. Moreover, the model of a fractional order heating furnace, for example, is more accurate than that of its traditional counterparts [10].

The accurate analysis of fractional order systems has been drawing an increasing attention [11–13]. However, the traditional analysis methods mainly depend on manual derivations, thus simulation methods have to be harnessed in engineering circumstances. Thus far, only the analytic solutions of some specific fractional order systems have been found. The conventional method of analyzing most fractional order systems is to determine their approximate results with the minimum number of errors by optimization algorithms [14–16]. Due to the complexity, accurate results cannot be attained through simulation methods, errors may be introduced. On the other hand, simulation of fractional calculus involves infinite sums, which consumes an enormous amount of memory.

Theorem proving is a formal technique which has been successfully applied to the accurate analysis of software and hardware systems [17]. Siddique [18] conducted a formal analysis of the Riemann-Liouville (RL) definition of fractional calculus and some related properties of the Gamma function using the higher-order logic method, and verified the fractional behaviors of capacitance and resistance.

In this paper, we present a formalization of a fractional order linear system in a higher-order logic theorem proving system. The rest of the paper is organized as follows. The definition of a fractional order linear system and its related properties are described in Section II. In Section III, the analytical framework for formally analyzing fractional order systems is presented. The formalization and verification of a fractional order linear system in HOL4 (a higher-order logic theorem prover) are presented in Section IV. In Section V, the specific fractional control system is verified using the proposed method and the correctness and effectiveness of this method are discussed. Finally, conclusions are drawn in Section VI.

2. Fractional order linear systems

The basic operator of a fractional calculus operator is ${}_aD_t^{\alpha}$ [19], where a and t are the lower and upper limits of the operator, and α

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is the calculus order.

$$ {}_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha} & R(\alpha) > 0 \\ 1 & R(\alpha) = 0 \\ \int_a^t (d\tau)^{(-\alpha)} & R(\alpha) < 0 \end{cases} \quad (1) $$

In Eq. (1), $R(\alpha)$ is the real part of α.

The fractional order linear system is the most basic type of fractional order system [20], whose general form is

$$ a_n D_t^{\beta_n} y(t) + a_{n-1} D_t^{\beta_{n-1}} y(t) + \cdots + a_1 D_t^{\beta_1} y(t) + a_0 D_t^{\beta_0} y(t) = u(t) \quad (2) $$

In Eq. (2), $u(t)$can be constructed using certain function and its fractional calculus:

$$ u(t) = b_m D_t^{\alpha_m} x(t) + b_{m-1} D_t^{\alpha_{m-1}} x(t) + \cdots + b_1 D_t^{\alpha_1} x(t) + b_0 D_t^{\alpha_0} x(t) \quad (3) $$

Similar to the integer order calculus, the fractional calculus has the following properties [21]:

(1) The operation ${}_0D_t^\alpha$ and integer order calculus produce the same result if $\alpha \in Z$. In addition, we have

$$ {}_aD_t^0 f(t) = f(t) \quad (4) $$

(2) The operator is linear. For the arbitrary constants of a and b, we have

$$ {}_aD_t^v [cf(t) + dg(t)] = c\, {}_aD_t^v f(t) + d\, {}_aD_t^v g(t) \quad (5) $$

(3) The fractional calculus operator satisfies the exponentiation multiplication law (superposition property):

$$ {}_aD_t^u [{}_aD_t^v f(t)] = {}_aD_t^{u+v} f(t) \quad (6) $$

In this paper, we focus on the case where the fractional order is a real number.

3. Formal analytical framework

A model framework of higher-order logic theorem proving based on fractional calculus is given in Fig. 1. The key contributions of the paper are the gray shaded boxes in the figure. They are the foundation of the formal analysis of a fractional order system in HOL4 [22]. We formalize the GL definition based on the real binomial coefficient. The starting point of fractional calculus analysis is a description of the fractional order system and the relative properties of the system. The target is to verify the correctness of the fractional order model based on the GL definition and the relative properties of the fractional order system.

According to the proposed framework, a formal model of the fractional order system is constructed in higher-order logic. For this purpose, the fractional calculus, based on the GL definition, is formalized in higher-order logic. The formalization of the GL definition is based on some mathematical theories, including real number and integer order calculus and the real binominal coefficient. Real number and integer order calculus were formalized by Harrison [23]. Real binomial coefficient was formalized by Shi [24]. Building on this work, this paper provides the formalization of the GL definition of fractional calculus, which is used in the higher-order logic modeling of the fractional order linear system as well as its verification. Second, the formalization of the fractional order linear system is conducted using the higher-order logic model constructed in the first step, namely, to express system properties as higher-order logic functions. Third, the formal verification of the above higher-order logic functions is performed in HOL4. Finally,

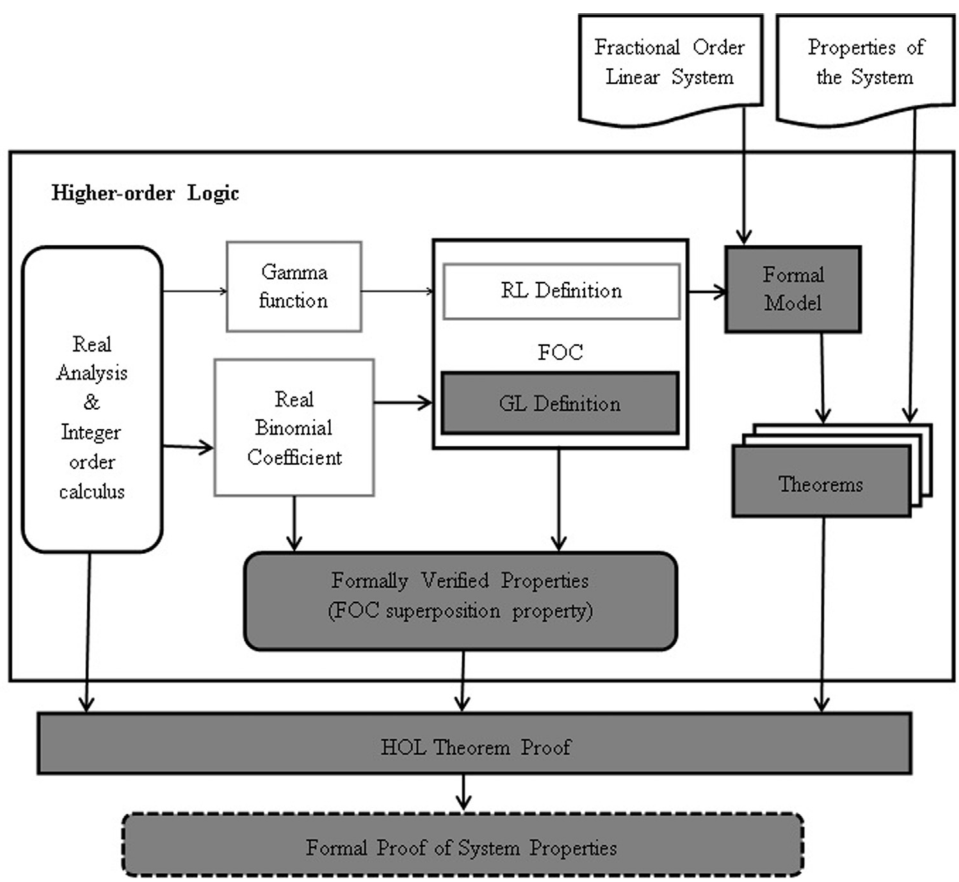

Fig. 1. Higher-order logic analytical framework based on fractional calculus.

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