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Research Article

An innovative fixed-pole numerical approximation for fractional order systems

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ABSTRACT

A novel numerical approximation scheme is proposed for fractional order systems by the concept of identification. An identical equation is derived firstly, from which one can obtain the exact state space model of fractional order systems. It reveals the nature of the approximation problem, and then provides an effective scheme to obtain the desired model. This research project also focuses on solving a knotty but crucial issue, i.e., the initial value problem of fractional order systems. The results generated by the study prove that it can reduce to the Caputo case by selecting some specific initial values. A careful simulation study is reported to illustrate the effectiveness of the proposed scheme. To exhibit the superiority clearly, the results are compared with that of the published fixed-pole finite model method.

1. Introduction

The idea of fractional order calculus has been known since the development of the regular integer order calculus. The initial research is associated with Leibniz and L'Hospital [1–3]. Although fractional order calculus has been existing for a long time, the development in this research field is rather slower than the integer order case. Nonetheless, the attention to fractional order calculus has been increasing in the control community since the 70s of last century. There are mainly two reasons for this boom. First, various physical practical materials and processes can be properly described by using fractional order calculus, which provides excellent tools to describe their properties [4,5]. Second, compared with the conventional (integer order) controllers, fractional order controllers exhibit their superiority in the transient performance, the robust ability and the design freedom [6–8]. Due to the tremendous efforts devoted by researchers, valuable results have been obtained on fractional order calculus. The details of the most recent advance in fractional order calculus can be found in some excellent monographs [9–11], review articles [4,5], and the references listed in the paper.

What can be concluded from the studies is that there are two essential characteristics of fractional order calculus. First, as a generalization of traditional integer order calculus, it can describe many practical plants more precisely in frequency domain, such as, using the magnitude–frequency characteristic curve with the slope of -20α dB/oct, $\alpha \in (0, 1)$. Second, it has the long range memory phenomenon. It can reflect and describe the effects caused by historical data on the system well. These characteristics bring some limitations in return. For example, fractional order calculus has weak singularity, which leads to the difficulty of solving the analytic solutions for fractional differential equations. Further, this also makes it difficult to simulate fractional order systems and implement fractional order controllers [10]. Therefore, a number of achievements in its alternative numerical methods grow vigorously [12–14]. Among these methods, there is a common procedure. That is, the fractional order calculus operators will be approximated by anyone of the methods first, then the relevant approximate results are further extended to fractional order systems.

In 2000, the noted Oustaloup method was proposed for fractional order differential operators, which is a recursive approximation algorithm [15]. In 2003, another approximation method for fractional order integral operators was developed [16]. These methods can achieve good approximate performance in both magnitude frequency and phase frequency characteristics. Yet, these methods do not have a good performance at the edge of the

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frequency range that is interested. In 2006, the Oustaloup method was modified, so that the approximate accuracy in the whole frequency range was improved [17]. Afterward, a chain of pertinent methods was proposed [18–20], which are all aimed to improve the approximate accuracy by different means. Until 2014, there came a breakthrough in the research that the focus of the study is turned from high approximate accuracy to low approximate order revolutionarily [21]. By using this proposed first-order parallel frequency model (FPFM) method, a lower order approximate model can be obtained without sacrificing the accuracy.

Although some aforementioned methods have been extensively used in many applications, there still exist several limitations in such methods. To break through these limitations, this paper derives a new scheme with the following improvements. (i) With the aids of the identification, we configure the zeros and poles of the approximation model's transfer function directly, while the published Oustaloup-based-methods have wrong ideas in this issue. They assume the Bode magnitude diagram as a polygonal line which is actually a smooth curve. Thus, as a result, there are no strict turning points which were treated as zeros and poles. (ii) The relevant study on the relationship between the pure integration feature of $\frac{1}{s^\alpha}$ and the order α is conducted and the implicit character is revealed. Hereafter, we provide a guidance on whether the first pole should be set as zero. As of today, it is the first time to consider this issue. (iii) In our design, zeros and poles are selected in complex domain while the existing methods set poles and zeros as real numbers. The extension of these ranges will result in the potent to achieve a higher approximation accuracy. (iv) The fixed-pole idea is introduced and applied in our design, which corresponds to lower approximate order and larger engineering applicability than that of varying-pole method. In other words, the proposed scheme does not only focus on the approximate accuracy but also focus on the implementation costs, which is the same as [21]. (v) The initial value problem is considered in the design procedure of the proposed method while the existing methods rarely focus on it. However, the initial value problem is indeed an inevitable issue of fractional order systems [22,23].

The rest of the content stated in this paper is organized as follows. Section 2 presents basic definitions and relevant facts for fractional order calculus. Additionally, the problem highlighted above is clearly defined. Section 3 establishes a novel fixed-pole numerical approximation scheme for fractional order systems. Section 4 provides four numerical examples to validate the wide applicability of the proposed method. Section 5 draws the conclusions.

2. Preliminaries

2.1. Fractional order calculus

Fractional order calculus is the generalization and unification of the traditional integer order calculus. There are many definitions related to the different fractional order derivatives, such as, Riemann–Liouville, Caputo and Grünwald–Letnikov. However, all of them have weak singularity that generates globality in space and long memory in time. As of today, there is no global consensus for the state space representation of a fractional order system, particularly on the process of initialization.

Note that these widely applicable fractional order derivative definitions are derived from Riemann–Liouville integral [9]. According to this definition, the α-th integral of function $u(t)\in\mathbb{R}$ can be expressed as

$$y(t)={}_{t_0}\mathcal{I}_t^{\alpha}(u(t))=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}u(\tau)\,\mathrm{d}\tau, \tag{1}$$

with the Gamma function $\Gamma(z)=\int_0^\infty x^{z-1}e^{-x}\,\mathrm{d}x$ and positive real α.

The Laplace transform of the fractional order integration operation can be formulated as

$$\mathcal{I}^{\alpha}(s)=\frac{\mathcal{L}[y(t)]}{\mathcal{L}[u(t)]}=\frac{1}{s^{\alpha}}. \tag{2}$$

This fractional order integrator $\mathcal{I}^{\alpha}(s)$ is the key element in the simulation process of the fractional order control systems. However, the realization of $\mathcal{I}^{\alpha}(s)$, either in analog or numerical form, is not a simple task. Especially, when one considers the initial conditions, it becomes much difficult than that of the integer order case.

2.2. The exact state space model

Theorem 1. *For any $s\in\mathbb{C}$ and $\alpha\in(0,1)$, the following equation holds:*

$$\int_0^\infty\frac{\mu_\alpha(\omega)}{s+\omega}\,\mathrm{d}\omega=\frac{1}{s^{\alpha}}, \tag{3}$$

where $\mu_\alpha(\omega)=\frac{\sin(\alpha\pi)}{\omega^{\alpha}\pi}$.

Proof. Due to the fact that $\frac{\pi}{\sin(\alpha\pi)}$ is a special value in complex analysis, (3) can be equivalently expressed as

$$\int_0^\infty\frac{1}{\omega^{\alpha}}\frac{1}{1+\frac{\omega}{s}}\frac{s^{\alpha}}{s}\,\mathrm{d}\omega=\frac{\pi}{\sin(\alpha\pi)}. \tag{4}$$

Defining a new integral variable $t=\frac{\omega}{s}$, then (4) can be transformed as

$$\int_0^\infty\frac{1}{t^{\alpha}}\frac{1}{1+t}\,\mathrm{d}t=\frac{\pi}{\sin(\alpha\pi)}. \tag{5}$$

At this point, we just need to prove the establishment of (5).

Recalling the definition and properties stated in Beta function [9], one can obtain

$$B(\alpha,1-\alpha)=B(1-\alpha,\alpha)=\int_0^1\tau^{-\alpha}(1-\tau)^{\alpha-1}\,\mathrm{d}\tau=\frac{\pi}{\sin(\alpha\pi)}. \tag{6}$$

If we define a new integral variable $t=\frac{\tau}{1-\tau}$, the above equation becomes

$$B(\alpha,1-\alpha)=\int_0^\infty\frac{1}{t^{\alpha}}\frac{1}{1+t}\,\mathrm{d}t, \tag{7}$$

and this completes the proof.□

By introducing the concept of transfer function, Lemma 1 can be obtained easily.

Lemma 1 (*Montseny [24]*). *The fractional order system $\mathcal{D}^{\alpha}y(t)=u(t)$ with $u(t)\in\mathbb{R}$ and $0<\alpha<1$ is actually a linear frequency distributed system. Its frequency distributed state $z(\omega,t)\in\mathbb{R}$ satisfies*

$$\frac{\partial z(\omega,t)}{\partial t}=-\omega z(\omega,t)+v(t) \tag{8}$$

and the output $y(t)$ of the fractional order integrator is the weighted integral

$$y(t)=\int_0^\infty\mu_\alpha(\omega)z(\omega,t)\,\mathrm{d}\omega. \tag{9}$$

Remark 1. In fact, because of the definition of transfer function, zero initial conditions should be guaranteed, i.e., $y(t)=0, t\le 0$.