ISA Transactions 68 (2017) 293–301

Contents lists available at ScienceDirect

ISA Transactions

journal homepage: www.elsevier.com/locate/isatrans

# Analysis of the optimal sampling rate for state estimation in sensor networks with delays ☆

Miguel Martínez-Rey, Felipe Espinosa *, Alfredo Gardel

*Electronics Department, Engineering School, University of Alcala, University Campus s/n, 28871 Alcalá de Henares, Madrid, Spain*

ARTICLE INFO

*Article history:*
Received 31 October 2016
Received in revised form
2 February 2017
Accepted 14 March 2017
Available online 28 March 2017

*Keywords:*
Optimal sampling
State estimation
Kalman filter
Riccati equation
Sensor network

ABSTRACT

When addressing the problem of state estimation in sensor networks, the effects of communications on estimator performance are often neglected. High accuracy requires a high sampling rate, but this leads to higher channel load and longer delays, which in turn worsens estimation performance. This paper studies the problem of determining the optimal sampling rate for state estimation in sensor networks from a theoretical perspective that takes into account traffic generation, a model of network behaviour and the effect of delays. Some theoretical results about Riccati and Lyapunov equations applied to sampled systems are derived, and a solution was obtained for the ideal case of perfect sensor information. This result is also interesting for non-ideal sensors, as in some cases it works as an upper bound of the optimisation solution.



## 1. Introduction

Research on Networked Control Systems (NCS) and wireless sensor networks is often aimed at optimising network resources [1]. One important area of research focuses on reducing the amount of messages transmitted through the network [2–4] in order to save power and avoid degradation of the communication channel caused by high occupation. Ironically, most of these studies assume ideal communications between the system elements.

The effects of non-ideal communication on estimation over sensor networks, due to packet delays and/or packet loss, and how to deal with them, have been extensively studied in isolation [5–10]. Additionally, the dependency between traffic load, message delay and packet loss has been studied using computer network models [11–13], which belong to an entirely different field of engineering.

In short, sensors create traffic, the traffic induces a network delay, and the delayed measurements have an impact on estimation performance. While these three effects are normally studied separately, they should be addressed in conjunction in order to design transmission policies that achieve the desired performance.

Bringing this idea back to the context of optimisation problems in sensor networks, the classical approach is to optimise a cost function where, among other things, a given cost is assigned to measurement transmissions [14–16]. This cost is set according to the undesired effects caused by transmitting too many samples. Usually, however, low occupancy is not the main objective. Often the real objective is to achieve high estimation accuracy, and measurement delay is simply a consequence and a hurdle that negatively affects the estimates.

In this classical approach, if we wish to obtain the best possible accuracy regardless of the amount of samples, we would favour the error cost over the transmission cost. However, this would not work because it would increase the transmission rate at the expense of increasing the network load, as more information from the sensors is required. The consequence would be higher congestion in the channel and longer sample delays, which in turn would lead to poor performance.

This paper studies the problem of determining an optimal sampling rate for estimation in terms of the Mean Squared Error (MSE), taking into account the traffic generated by the sensors and the negative effects of the delayed measurements caused by this traffic. As a first approach to the problem, we assume a linear system and periodic sampling. This is reasonable since studies of the infinite horizon sensor scheduling problem have concluded that the solutions are periodic [15]. The network topology is obviated and we assume a centralised Kalman filter, which gives an optimal estimation.

☆This work was supported by the Spanish Ministry of Economy and Competitiveness through the ALCOR Project (DPI2013-47347-C2-1-R), as well as the grant program FPI UAH 2012 funded by University of Alcalá.

* Corresponding author.
*E-mail addresses:* felipe.espinosa@uah.es (F. Espinosa), alfredo.gardel@uah.es (A. Gardel), miguel.martinez@depeca.uah.es (M.Martínez-Rey).

http://dx.doi.org/10.1016/j.isatra.2017.03.007
0019-0578/

### 1.1. Notation

The following notation is used in this paper: $\mathbb{R}^+$ is the set of all positive real numbers. $X'$ is the transpose of $X$ and tr$X$ is the matrix trace. The ∘sign denotes function composition, i.e. $f \circ g(x) = f\,(g(x))$ . $X > 0$ means $X$ is positive-definite. If $\geq$ is used instead, then it is positive-semidefinite. Accordingly, $X > Y$ means $X - Y > 0$.

## 2. Background

The plant model of the system we want to estimate is given by a set of linear differential equations

$$\dot{x}(t) = Ax(t) + w(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$ is the system matrix and $w(t) \in \mathbb{R}^n$ is the system noise. The system noise is modelled as a Gaussian process with zero mean and covariance matrix

$$E\left[\,w(t)w(\tau)'\,\right] = S\delta(t - \tau) \tag{2}$$

where $S \in \mathbb{R}^{n\times n}$, $S \geq 0$ and $\delta(\cdot)$ is the Dirac delta. The system model (1) may include an input term $Bu$, but since this does not affect computation of the estimation covariance matrices, we will consider a system with no inputs.

### 2.1. Sampled systems

The sensor network obtains measurements $y_k \in \mathbb{R}^m$ at time instants $t_k = kT$, where $T$ is the sampling time, $T \in \mathbb{R}^+$, and $k \in \mathbb{N}$. We assume that this sampling time can be adjusted at the sensor, and that it has a minimum value given by technological limitations.

The equation of the output is

$$y_k = Hx(kT) + v_k \tag{3}$$

where $H \in \mathbb{R}^{m\times n}$ is the output matrix and $v_k \in \mathbb{R}^m$ is the measurement noise. The noise is assumed to be an uncorrelated Gaussian process whose covariance matrix is

$$E\left[\,v_k v_k'\,\right] = R; \quad R \in \mathbb{R}^{m\times m},\; R > 0\ . \tag{4}$$

According to the value of $T$, the system must be transformed into a discrete-time system. The solution of the differential equation (1), given an initial condition $x(t_0) = x_0$, is

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}w(\tau)d\tau\ . \tag{5}$$

The above equation is equivalent to

$$x(t_0 + \Delta t) = F_{\Delta t}x(t_0) + \bar{w}_{\Delta t}(t_0) \tag{6}$$

where

$$F_{\Delta t} = e^{A\Delta t} \tag{7}$$

is the transition matrix and

$$\bar{w}_{\Delta t}(t_0) = \int_{t_0}^{t_0+\Delta t} e^{A(t_0+\Delta t-\tau)}w(\tau)d\tau \tag{8}$$

is the equivalent discrete-time noise.

Hence, (6) can be converted to a recursive equation using $\Delta t = T$. This yields the difference equation of a discrete system for any chosen $T$.

$$x((k + 1)T) = F_T x(kT) + \bar{w}_T(kT) \tag{9}$$

As $w$ is stationary, so is $\bar{w}_T$, and its covariance matrix is

$$E\left[\,\bar{w}_T(t)\bar{w}_T(t)'\,\right] = Q_T = \int_0^T e^{A\tau}Se^{A'\tau}\,d\tau\ . \tag{10}$$

For a comprehensive discussion on how to compute the discrete system matrix $F_T$ using numerical methods, see [17]. A fairly simple method for computing $Q_T$ exists [18], which transforms the problem into solving Lyapunov equations.

Once we have a discrete-time system, we can obtain an estimation of the state vector $\hat{x}$ using a discrete-time Kalman filter, which is the optimal estimator in terms of the MSE. When a sample is received, the estimation error covariance matrix $P = E[(x - \hat{x})(x - \hat{x})']$ is updated.

Let $g$ be the operator that performs the measurement update on $P$:

$$P^+(kT) = g\left(P^-(kT)\right) \tag{11}$$

where $P^-$ and $P^+$ are the *a priori* and the *a posteriori* values of the covariance. Function $g$ is defined as

$$g(X) = X - XH'\left(HXH' + R\right)^{-1}HX\,. \tag{12}$$

In between measurements, $P$ evolves according to the Continuous-time Differential Lyapunov Equation (CDLE).

$$\dot{P}(t) = AP(t) + P(t)\,A' + S \tag{13}$$

Again, the solution of this equation, for a given initial condition $P_0 = P(t_0)$, is

$$P(t) = F_{t-t_0}P_0F_{t-t_0}' + Q_{t-t_0}\,. \tag{14}$$

Let operator $h_{\Delta t}$ be the propagation of the covariance during prediction over $\Delta t$ seconds.

$$h_{\Delta t}(X) = F_{\Delta t}XF_{\Delta t}' + Q_{\Delta t} \tag{15}$$

A nice property of this operator is that

$$h_{t_1+t_2}(X) = h_{t_1}\circ h_{t_2}(X) = h_{t_2}\circ h_{t_1}(X). \tag{16}$$

Then, the prediction stage of the estimator is computed with the formula

$$P^-\left((k + 1)T\right) = h_T\left(P^+(kT)\right). \tag{17}$$

The steady-state estimation error covariance matrix $\bar{P}_T$ of the filter is the solution of the equation $\bar{P}_T = h_T\circ g\left(\bar{P}_T\right)$, yielding the Discrete-time Algebraic Riccati Equation (DARE).

$$\bar{P}_T = F_T\bar{P}_TF_T' - F_T\bar{P}_TH'\left(H\bar{P}_TH' + R\right)^{-1}H\bar{P}_TF_T' + Q_T \tag{18}$$

Having determined $\bar{P}_T$, we can also define

$$\bar{P}_T^+ = g\left(\bar{P}_T\right) \tag{19}$$

as the *a posteriori* steady-state estimation error covariance matrix, which is in turn the solution of the equation $\bar{P}_T^+ = g\circ h_T\left(\bar{P}_T^+\right)$.

The solution $\bar{P}_T$ of the DARE (18) corresponds to a specific sampling period $T$ since the transition matrix and the equivalent discrete noise covariance must be computed for each value of $T$. This means that we can think of $\bar{P}_T$ as a function of $T$. Such a function does not have a mathematical expression in closed form, and to the best of our knowledge, its properties have not been studied in the literature.

## 3. Problem description

It is usual to assume that increasing the sensor sampling rate will improve the performance of the remote estimator. However,