ISA Transactions 64 (2016) 77–85

Contents lists available at ScienceDirect

ISA Transactions

journal homepage: www.elsevier.com/locate/isatrans

Practical stability analysis of fractional-order impulsive control systems

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ARTICLE INFO

Article history:
Received 17 January 2016
Received in revised form
18 April 2016
Accepted 17 May 2016
Available online 9 June 2016
This paper was recommended for publication by Mohammad Haeri.

Keywords:
Practical stability
Fractional differential systems
Impulsive control

ABSTRACT

In this paper we obtain sufficient conditions for practical stability of a nonlinear system of differential equations of fractional order subject to impulse effects. Our results provide a design method of impulsive control law which practically stabilizes the impulse free fractional-order system.

1. Introduction

Impulsive control arises naturally in a wide variety of applications. Various good impulsive control approaches for integer-order systems have been proposed in many fields such as neural networks [1,2], ecosystems management [3,4], epidemic models [5], financial models [6,7], etc. There are many cases where impulsive control can give better performance than continuous control. Sometimes even only impulsive control can be used for control purpose. The centrality of impulsive control strategies for theory and applications is witnessed by the current persistency of new contributions in this topic of interest [8–10].

On the other hand, fractional-order models are found to be more adequate than integer-order models, and there has been a significant development in the theory of fractional differential systems (FDSs) in the last years. For examples and details, see [11–13].

The explosion in research within the FDSs setting led to new developments in their qualitative theories. Also, in relation to the mathematical simulation in chaos, fluid dynamics and many physical systems, only relatively recently impulsive FDSs have started to receive an increasing interest [14–18]. Fractional calculus was introduced to the stability analysis of such systems, where integer-order methods were extended to fractional-order dynamic systems. See, for example, [19] and the references therein. However, the studies of impulsive FDSs mainly focus on the stability or asymptotic stability in the Lyapunov sense. In addition, few theoretical studies on impulsive synchronization and control of fractional-order systems are reported in the literature. The stability of impulsive fractional-order systems is investigated by employing Gronwal–Bellman's inequality in [20], and an impulsive synchronization criterion of fractional-order chaotic systems is obtained. In [21], a pinning impulsive control scheme is adopted to investigate the synchronization of fractional complex dynamical networks. The paper [22] studies optimal relaxed controls and relaxation of nonlinear fractional impulsive evolution equations. The paper [23] is concerned with feedback control systems governed by fractional impulsive evolution equations involving Riemann–Liouville derivatives in reflexive Banach spaces. The authors of [24] proposed an impulsive control scheme for fractional-order chaotic systems which is based on the Takagi–Sugeno fuzzy model and linear matrix inequalities. However, in spite of the great possibilities for applications, the stability and control theories of impulsive FDSs have not yet been fully developed and this paper's aim is mainly to fill the gap.

One of the most important aspects of the stability theory of differential equations is the so-called practical stability. The notion of practical stability of dynamical systems was first discussed by Lasalle and Lefshetz [25] in 1960s and since then a great progress has been made [26–29]. The main problem in the theory of practical stability consists of studying the solutions of systems of differential equations close to a certain state, given in advance the domain where the initial conditions change, and the domain where the solutions should remain when the independent variable changes over a fixed interval (finite or infinite). The desired state of a system may be unstable in the sense of Lyapunov and yet a

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http://dx.doi.org/10.1016/j.isatra.2016.05.012
0019-0578/

solution of the system may oscillate sufficiently near this state that its performance is acceptable. As such, practical stability and the Lyapunov stability are quite independent concepts, and, in general, neither imply nor exclude each other. In some cases, though a system is stable or asymptotically stable in the Lyapunov sense, it is actually useless in practice because of undesirable transient characteristics (e.g., the stability domain or the attraction domain is not large enough to allow the desired deviation to cancel out). The notion of practical stabilization is of a significant importance in scientific and practical engineering problems [30,31]. For example, it is very useful in estimating the worst-case transient and steady-state responses and in verifying pointwise in time constraints imposed on the state trajectories.

However, very little work has been done in practical stability analysis of fractional order system. The initial time difference practical stability has been investigated in terms of two measures for FDSs without impulsive perturbations in [32]. To the best of our knowledge, there has not been any work so far considering the practical stability of fractional impulsive control systems, which is very important in theories and applications and also is a very challenging problem.

In this paper, motivated by the above considerations, we generalize the concept of the practical stability to an impulsive control system of fractional order with Caputo fractional derivative. Using the fractional comparison principle proved in [19], we investigate the effect of the impulses on the practical stabilization of the system under consideration. Scalar and vector Lyapunov-like functions are also used to derive practical stability criteria for the impulsive control fractional differential system. Applications to linear and non-linear systems are discussed to illustrate the theory.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $||\cdot||$, Ω be an open set in $\mathbb{R}^n$ containing the origin, and let $\mathbb{R}_+ = [0,\infty)$. Let $t_0 \in \mathbb{R}_+$.

Definition 2.1 (*[12,19]*). For any $t \ge t_0$, the *Caputo fractional derivative of order q*, $0<q<1$, with the lower limit t_0 for a function $l \in C^1[[t_0,b],\mathbb{R}^n]$, $b>t_0$, is defined as

$$ {}_{t_0}^{\;c}D_t^q l(t) = \frac{1}{\Gamma(1-q)}\int_{t_0}^{t}\frac{l'(s)}{(t-s)^q}ds. $$

Here and in what follows Γ denotes the Gamma function.

We consider the following fractional-order impulsive control system

$$ {}_{t_0}^{\;c}D_t^q x(t) = f(t,x(t)) + \eta(t), \quad t>t_0, \tag{2.1} $$

where

$$ \eta(t) = \sum_{k=1}^{\infty} I_k(x(t))\delta(t-t_k), \tag{2.2} $$

is the control input, $\delta(t)$ is the Dirac impulsive function with discontinuity points

$$ t_0<t_1<t_2<\cdots<t_k<\cdots $$

and $\lim_{k\to\infty} t_k=\infty$, $x:[t_0,\infty)\to\mathbb{R}^n$; $f:[t_0,\infty)\times\Omega\to\mathbb{R}^n$, $I_k:\Omega\to\mathbb{R}^n$, $k=1,2,\dots$.

Note that [14–18,33] from (2.2), $\eta(t)=0$ for $t\neq t_k$, $k=1,2,\dots$. Then, we have

$$ x(t_k+h)-x(t_k) = \frac{1}{\Gamma(q)}\int_{t_0}^{t_k+h}(t_k+h-s)^{q-1}[f(s,x(s))+\eta(s)]ds $$
$$ -\frac{1}{\Gamma(q)}\int_{t_0}^{t_k}(t_k-s)^{q-1}[f(s,x(s))+\eta(s)]ds $$
$$ =\frac{1}{\Gamma(q)}\int_{t_k}^{t_k+h}(t_k+h-s)^{q-1}[f(s,x(s))+\eta(s)]ds $$
$$ +\frac{1}{\Gamma(q)}\int_{t_0}^{t_k}\left((t_k+h-s)^{q-1}-(t_k-s)^{q-1}\right)[f(s,x(s))+\eta(s)]ds, $$

where $h>0$ is sufficiently small. As $h\to 0^+$, we obtain

$$ \Delta x(t_k) = x(t_k^+)-x(t_k) = I_k(x(t_k)), $$

where $x(t_k^+)=\lim_{h\to 0^+}x(t_k+h)$.

The controller $\eta(t)$ has an effect on sudden changes in the state of (2.1) at the time instants t_k, i.e., $\eta(t)$ is an impulsive control of (2.1). The corresponding closed-loop nonlinear delayed equation of (2.1) under impulsive control (2.2) is given by

$$ \begin{cases} {}_{t_0}^{\;c}D_t^q x(t) = f(t,x(t)), & t\neq t_k,\\ \Delta x(t_k) = I_k(x(t_k)), & k=1,2,\dots, \end{cases} \tag{2.3} $$

where $t_k<t_{k+1}<\cdots$, $(k=1,2,\dots)$ are the moments of impulsive perturbations due to which the state $x(t)$ changes from the position $x(t_k)$ into the position $x(t_k^+)$; I_k are functions, which characterize the magnitude of the impulse effect at the moments t_k.

Let $x_0\in\Omega$. Denote by $x(t)=x(t;t_0,x_0)$ the solution of system (2.3), satisfying the initial condition

$$ x(t_0^+;t_0,x_0)=x_0. \tag{2.4} $$

We suppose that the functions f and I_k, $k=1,2,\dots$, are smooth enough on $[t_0,\infty)\times\Omega$ and Ω, respectively, to guarantee existence, uniqueness and continuability of the solution $x(t)=x(t;t_0,x_0)$ of the initial value problem (IVP) (2.3), (2.4) on the interval $[t_0,\infty)$ for each $x_0\in\Omega$ and $t\ge t_0$. The solutions $x(t;t_0,x_0)$ are, in general, piecewise continuous functions with points of discontinuity of the first type at which they are left continuous, that is, at the moments t_k, $k=1,2,\dots$, the following relations are satisfied [19]:

$$ x(t_k^-)=x(t_k) \quad\text{and}\quad x(t_k^+)=x(t_k)+I_k(x(t_k)). $$

For the basic theory on impulsive differential equations, the reader is referred to [34,35] and the references cited therein.

We shall introduce definitions of practical stability of system (2.3) which are analogous to the definitions given in [26].

Definition 2.2. The system (2.3) is said to be:

(a) *practically stable* with respect to (λ,A), if given (λ,A) with $0<\lambda<A$, we have $||x_0||<\lambda$ implies $||x(t;t_0,x_0)||<A$, $t\ge t_0$ for some $t_0\in\mathbb{R}_+$;
(b) *uniformly practically stable* with respect to (λ,A), if (a) holds for every $t_0\in\mathbb{R}_+$;
(c) *practically asymptotically stable* with respect to (λ,A), if (a) holds and $\lim_{t\to\infty}||x(t;t_0,x_0)||=0$.

For fractional-order systems we will introduce the new notion of practical Mittag–Leffler stability.

Definition 2.3. The system (2.3) is said to be *practically Mittag–Leffler stable* with respect to (λ,A), if given (λ,A) with $0<\lambda<A$, we have $||x_0||<\lambda$ implies

$$ ||x(t;t_0,x_0)|| \le \{AE_q[-\alpha(t-t_0)^q]\}^\beta, \quad t\ge t_0, $$

for some $t_0\in\mathbb{R}_+$, where E_q is the corresponding Mittag–Leffler function, $\alpha,\beta>0$.

Remark 2.1. The Mittag–Leffler stability notions for fractional-order systems are analogous to the exponential stability notions