Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 196 (2007) 1178–1192
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Rigid body concept for geometric nonlinear analysis of 3D frames, plates and shells based on the updated Lagrangian formulation

Y.B. Yang *, S.P. Lin, C.S. Chen

Department of Civil Engineering, National Taiwan University, Taipei 10617, Taiwan

Received 26 October 2005; received in revised form 18 July 2006; accepted 26 July 2006

Abstract

In the nonlinear analysis of elastic structures, the displacement increments generated at each incremental step can be decomposed into two components as the *rigid displacements* and *natural deformations*. Based on the updated Lagrangian (UL) formulation, the geometric stiffness matrix $[k_g]$ is derived for a 3D rigid beam element from the virtual work equation using a rigid displacement field. Further, by treating the three-node triangular plate element (TPE) as the composition of three rigid beams lying along the three sides, the $[k_g]$ matrix for the TPE can be assembled from those of the rigid beams. The idea for the UL-type incremental-iterative nonlinear analysis is that if the rigid rotation effects are fully taken into account at each stage of analysis, then the remaining effects of natural deformations can be treated using the small-deformation linearized theory. The present approach is featured by the fact that the formulation is *simple*, the expressions are *explicit*, and *all kinds* of actions are considered in the stiffness matrices. The robustness of the procedure is demonstrated in the solution of several benchmark problems involving the postbuckling response.

Keywords: Beam; Geometric nonlinear analysis; Plate; Postbuckling; Rigid element; Shell

1. Introduction

The nonlinear analysis of elastic structures is usually conducted in an incremental-iterative way based on the three configurations: the initial configuration C_0, last calculated configuration C_1, and current deformed configuration C_2, as indicated in Fig. 1. In a step-by-step nonlinear analysis, we are interested in the behavior of the structure during the incremental step from C_1 to C_2. The deformations occurring within each incremental step are assumed to be *small*, but the displacements accumulated for all incremental steps can be very large. The concept to be presented herein is based on the *updated Lagrangian* (UL) formulation, in that all quantities of the structure are expressed with reference to the last configuration C_1.

The displacement increments generated at each incremental step of an elastic nonlinear analysis can be composed into two components as the *rigid displacements* and *natural deformations* [1,2]. For most structures encountered in practice, the rigid component constitutes a much larger portion of the displacement increments at each incremental step with respect to the deformational component. For a UL-type incremental-iterative analysis, the idea is that if the rigid rotation effects for elements with *initial forces* (or stresses) are fully taken into account at each stage of analysis, then the remaining effects of natural deformations can be treated using the small-deformation linearized theory.

Concerning the incremental-iterative procedure, distinction should be made between the *predictor* and *corrector* stages [3,4]. The predictor relates to solution of the displacement increments $\{U\}$ of the structure for given load increments $\{P\}$ based on the structural equation $[K]\{U\} = \{P\}$. This stage determines the *trial* direction of iteration of the structure in the load–deflection space and

* Corresponding author. Tel.: +886 2 3366 4245; fax: +886 2 2362 2975.
E-mail address: ybyang@ntu.edu.tw (Y.B. Yang).

0045-7825/$ - see front matter
doi:10.1016/j.cma.2006.07.013

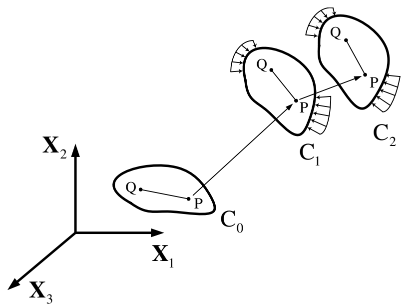

Fig. 1. Motion of body in three-dimensional space.

thus affects the number of iterations or speed of convergence. For this reason, the stiffness matrix $[K]$ used in the structural equation need *not* be exact, but should be kept *rigid-body qualified* to avoid convergence to incorrect directions. In the UL formulation, the corrector refers to recovery of the force increments $\{_2f\}$ at C_2 from the displacement increments $\{u\}$ made available through the structural displacement increments $\{U\}$, and the superimposition of these force increments with the initial nodal forces $\{^1_1f\}$ following the rigid body rule [5,6] for obtaining the total element forces $\{^2_2f\}$ at C_2.

In this paper, a rigid-body qualified geometric stiffness matrix $[k_g]$ will be derived for the 3D beam element from the virtual work equation by assuming the displacement field to be of the rigid type. Such an element is referred to as the *rigid element*. To the knowledge of the authors, no similar elements were presented by other scholars to explicitly accommodate the rigid behaviors of structures. For the 3D beam, the initial surface tractions may generate some moment terms upon 3D rotations during the incremental step from C_1 to C_2, commonly known as the moments induced by the *semitangential* torques and *quasi-tangential* bending moments [1,7]. Naturally, all such terms should be included in the virtual work formulation for the 3D beam. The other issue to be considered for the space frames is the equilibrium of angled joints in the *rotated configuration* C_2, rather than in C_1. Based on such a consideration, only the *symmetric* part of the geometric stiffness matrix of each element has to be retained in the structural stiffness matrix, as the antisymmetric parts of all the elements meeting at the same joint cancel out with each other [6,7].

As for the analysis of plate/shell problems, a *triangular plate element* (TPE) with three translational and three rotational degrees of freedom (DOFs) at each of the three tip nodes will be considered, for its compatibility with the 12-DOF beam element derived above. Since the *rigid body behavior* of each finite element is solely determined by its external shape or nodal DOFs, the geometric stiffness matrix for the TPE is derived by treating the TPE as the composition of three *rigid beams* lying along the three sides. The geometric stiffness matrix so derived is explicit and capable of dealing with all kinds of in-plane and out-plane actions.

For a review of related works on geometric nonlinear analysis of structures, Ref. [8] may be consulted, in which a total of 122 papers were cited. The purpose of this paper is not to review any related works. Rather, efforts will be focused on application of the rigid body concept and derivation of rigid-body qualified geometric stiffness matrices $[k_g]$ for the 3D beam element and TPE. The elastic stiffness matrices $[k_e]$ adopted are those readily available in the literature, namely, the elastic stiffness matrix $[k_e]$ adopted for the 3D beam element is the one commonly used [6,9], and the elastic stiffness matrix $[k_e]$ adopted for the TPE is constructed as the composition of Cook's plane hybrid element for membrane actions [10] and the *hybrid stress model* (HSM) of Batoz et al. for bending actions [11]. For the sake of brevity, repetition of relevant derivations is kept to the minimum.

2. Theory of three-dimensional beams

Before we proceed to derive the rigid element for the 3D beam, a summary of the theory for the 3D beam with bisymmetric solid cross-sections is first given. The beam element considered has a total of 12 DOFs as shown in Fig. 2, with x denoting the centroidal axis and (y,z) the two principal axes of the cross-section. Based on the UL formulation, the virtual work equation for a 3D beam at C_2, but with reference to C_1, can be expressed in a linearized form as [6]:

$$\begin{aligned}&\int_V (E_1e_{xx}\delta_1e_{xx} + 4G_1e_{xy}\delta_1e_{xy} + 4G_1e_{xz}\delta_1e_{xz})\,\mathrm{d}V\\&\quad + \int_V ({}^1\tau_{xx}\delta_1\eta_{xx} + 2{}^1\tau_{xy}\delta_1\eta_{xy} + 2{}^1\tau_{xz}\delta_1\eta_{xz})\,\mathrm{d}V\\&\quad = {}^2_1R - {}^1_1R, \end{aligned}\tag{1}$$

where E and G denote the elastic and shear modulus, respectively, V is the volume of the element, and the factors of 4 and 2 are added to account for the symmetry of shear strains, i.e., $_1e_{xy} = {_1e_{yx}}$, $_1e_{xz} = {_1e_{zx}}$, $_1\eta_{xy} = {_1\eta_{yx}}$, $_1\eta_{xz} = {_1\eta_{zx}}$, $({}^1\tau_{xx}, {}^1\tau_{xy}, {}^1\tau_{xz})$ are the initial (Cauchy) axial and shear stresses, and δ denotes the variation of the quantity following. The linear and nonlinear components of the strain increments can be expressed with reference to C_1 as

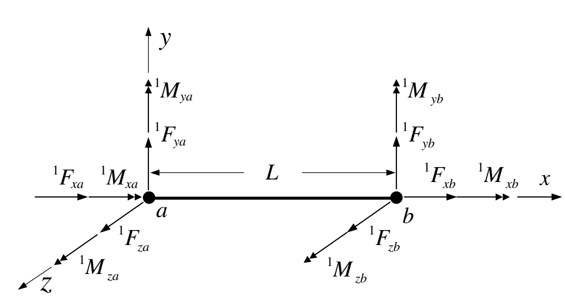

Fig. 2. Three-dimensional beam element.