Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 196 (2007) 1261–1277
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Axisymmetric pressure boundary loading for finite deformation analysis using p-FEM

Zohar Yosibash [a,*], Stefan Hartmann [b], Ulrich Heisserer [c], Alexander Düster [c], Ernst Rank [c], Mordechai Szanto [a]

[a] Pearlstone Center for Aeronautical Studies, Department of Mechanical Engineering, Ben-Gurion University of the Negev, Beer-Sheva, Israel
[b] Institute of Mechanics, University of Kassel, Germany
[c] Lehrstuhl für Bauinformatik, Fakultät für Bauingenieur-und Vermessungswesen, Technische Universität München, Germany

Received 4 May 2006; received in revised form 31 August 2006; accepted 7 September 2006

Abstract

Follower loads, i.e. loads which depend on the boundary displacements by definition, frequently occur in finite deformation boundary-value problems. Restricting to axisymmetrical applications, we provide analytical and numerical solutions for a set of problems in compressible Neo-Hookean materials so to serve as benchmark problems for verifying the accuracy and efficiency of various FE methods for follower load applications. Thereafter, the weak formulation for the follower-load in 3-D domain is reduced to an axisymmetrical setting, and, subsequently, consistently linearized in the framework of p-FEMs, exploiting the blending function mapping techniques. The set of axisymmetric benchmark solutions is compared to numerical experiments, in which the results obtained by a p-FEM code are compared to these obtained by a state-of-the-art commercial h-FEM code and to the "exact" results. These demonstrate the efficiency and accuracy of p-FEMs when applied to problems in finite deformations with follower loads.

Keywords: p-FEM; Follower load; Finite strains; Hyper-elasticity; Axisymmetry

1. Introduction

Problems of continuum mechanics are usually associated with large deformations and large strains, i.e. the length, shape and orientation of the domain's boundary changes during these highly non-linear loading processes. Cold Iso-static Pressing (CIP) of metal powders is a typical example of such problems, in which the tractions on the boundary and their directions, due to the applied pressure, change according to the deformation. So far, verification examples are unavailable in general, and numerical approximations, usually by finite element methods (FEMs), are sought.

Follower-loads have been addressed for over three decades and implemented in various low-order FEMs (also known as h-FEMs), see e.g. [1–4]. However, to the best of our knowledge no analytical solutions for finite deformations are available for compressible material models, which are commonly used in FE codes, and that may serve as benchmark problems for verification of the numerical solvers. For this reason, the first step herein is to derive simple analytical and comparable solutions for axisymmetric problems which may serve as benchmark problems to assess the accuracy and efficiency of numerical approximations. In the second step, we concentrate our attention on follower loads (also known as "deformation-dependent", or "path-following" loads) in the framework of high-order FEMs (p-FEMs) [5,6], which have been shown to perform well for finite deformations analyses [7]. Following [3], the

* Corresponding author. Tel.: +972 8 6477103; fax: +972 8 6477101.
E-mail addresses: zohary@bgu.ac.il (Z. Yosibash), stefan.hartmann@uni-kassel.de (S. Hartmann), heisserer@bv.tum.de (U. Heisserer), duester@bv.tum.de (A. Düster), rank@bv.tum.de (E. Rank), mszan@ bgu.ac.il (M. Szanto).

0045-7825/$ - see front matter
doi:10.1016/j.cma.2006.09.006

weak formulation associated with the follower-load in three-dimensional domain is reduced to an axisymmetrical setting, and, subsequently, consistently linearized in the framework of p-FEMs, exploiting the blending function mapping techniques. A set of axisymmetric numerical experiments is then addressed, in which the results obtained by a p-FEM analysis are compared to these obtained by a state-of-the-art commercial h-FEM code and to "exact" results. These demonstrate the efficiency and accuracy of p-FEMs when applied to problems in finite deformations with follower loads.

We start with notations and by deriving analytical solutions to finite deformation axisymmetric problems in Section 2. A compressible hyper-elastic material described by a Neo-Hooke type constitutive relation and loaded by pressure boundary condition (follower load) is considered. For these example problems we provide analytical solutions and numerical approximations computed by the shooting method for solving the underlying two-point boundary-value problem. In the sequel, these solutions serve as benchmark examples. Section 3 compiles the theoretical basis for the implementation of pressure loads into a FE code. In this section, we derive the weak form associated with the follower load for a three-dimensional domain, and present the consistent linearization of it. This results in two terms – a non-linear form, and a bi-nonlinear form. These two terms are restricted to axisymmetric domains. The formulation for p-axisymmetric elements is then provided in Section 4. We start this section by briefly presenting the special features of p-FE methods followed by a more detailed discussion on the implementation of the follower loads in p-FE framework. The iterative scheme for the solution of the non-linear problem is discussed. The efficiency and accuracy of our implementation is demonstrated in Section 5 on five example problems, and compared to the commercial h-FE code Abaqus.[1]

2. Verification examples in axisymmetric domains

In the following, we generate analytical/semi-analytical solutions for axisymmetric domains, based on constitutive assumptions of compressibility and isotropy. A brief description of notations for finite strain hyper-elasticity is provided followed by derivation of several analytical/semi-analytical solutions that serve as benchmarks against which the FE implementation can be verified.

The basic quantity is the deformation gradient $\boldsymbol{F} = \mathrm{Grad}\boldsymbol{\varphi}(\boldsymbol{X},t) = \frac{\partial\varphi^k(X^1,X^2,X^3,t)}{\partial X^K}\boldsymbol{g}_i \otimes \boldsymbol{G}^K$, where $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X},t)$ defines the placement of the material point $\boldsymbol{X}$ at time t. X^K, $k = 1,2,3$, are material (curvilinear) coordinates, $\boldsymbol{g}_i$ are tangent and $\boldsymbol{G}^K$ gradient vectors in current and the initial configurations. Since the most general strain-energy function for isotropic hyper-elastic material $\psi(\boldsymbol{C}) = \Psi(\mathbf{I}_C,\mathbf{II}_C,\mathbf{III}_C)$ or $\psi(\boldsymbol{b}) = \Psi(\mathbf{I}_b,\ \mathbf{II}_b,\mathbf{III}_b)$ depends on the invariants of the right Cauchy–Green tensor $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$, or the left Cauchy–Green tensor $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$, we define

$$\mathbf{I}_C = \mathrm{tr}\boldsymbol{C}, \quad \mathbf{II}_C = \frac{1}{2}((\mathrm{tr}\boldsymbol{C})^2 - \mathrm{tr}\boldsymbol{C}^2),$$
$$\mathbf{III}_C = \det\boldsymbol{C} = (\det\boldsymbol{F})^2 =: J^2, \tag{1}$$
$$\mathbf{I}_b = \mathrm{tr}\boldsymbol{b}, \quad \mathbf{II}_b = \frac{1}{2}((\mathrm{tr}\boldsymbol{b})^2 - \mathrm{tr}\boldsymbol{b}^2),$$
$$\mathbf{III}_b = \det\boldsymbol{b} = (\det\boldsymbol{F})^2 =: J^2, \tag{2}$$

where $\mathrm{tr}\boldsymbol{C} = C^N_N$ (equivalently, $\mathrm{tr}\boldsymbol{b} = b^n_n$) symbolizes the trace operator. In the current configuration the Cauchy stress tensor $\boldsymbol{\sigma}$ reads

$$\boldsymbol{\sigma} = \frac{2\rho_R}{J}\frac{\mathrm{d}\psi(\boldsymbol{b})}{\mathrm{d}\boldsymbol{b}}\boldsymbol{b} = \frac{2\rho_R}{J}\boldsymbol{b}\frac{\mathrm{d}\psi(\boldsymbol{b})}{\mathrm{d}\boldsymbol{b}} = \alpha_0\boldsymbol{I} + \alpha_1\boldsymbol{b} + \alpha_2\boldsymbol{b}^2 \tag{3}$$

where ρ_R is the density in initial configuration and

$$\alpha_0 = 2\rho_R\frac{\partial\Psi}{\partial\mathbf{III}_b}\mathbf{III}_b^{1/2},$$
$$\alpha_1 = 2\left(\rho_R\frac{\partial\Psi}{\partial\mathbf{I}_b} + \rho_R\frac{\partial\Psi}{\partial\mathbf{II}_b}\mathbf{I}_b\right)\mathbf{III}_b^{-1/2},$$
$$\alpha_2 = -2\rho_R\frac{\partial\Psi}{\partial\mathbf{II}_b}\mathbf{III}_b^{-1/2}. \tag{4}$$

Here, use is made of

$$\frac{\mathrm{d}\mathbf{I}_b}{\mathrm{d}\boldsymbol{b}} = \boldsymbol{I}, \quad \frac{\mathrm{d}\mathbf{II}_b}{\mathrm{d}\boldsymbol{b}} = \mathbf{I}_b\boldsymbol{I} - \boldsymbol{b}, \quad \frac{\mathrm{d}\mathbf{III}_b}{\mathrm{d}\boldsymbol{b}} = \mathbf{III}_b\boldsymbol{b}^{-1} = \mathrm{adj}\boldsymbol{b}, \tag{5}$$

which result from the application of the chain rule. The above relations are valid for any isotropic hyper-elastic material. We consider herein the simplest strain-energy function (SEF) of Neo-Hooke type:

$$\rho_R\psi(\boldsymbol{C}) = \frac{K}{2}(J-1)^2 + c_{10}(\mathbf{I}_{\overline{C}} - 3) \tag{6}$$
$$= \frac{K}{2}(\mathbf{III}_C^{1/2} - 1)^2 + c_{10}(\mathbf{I}_C\mathbf{III}_C^{-1/3} - 3). \tag{7}$$

$\mathbf{I}_{\overline{C}} = \mathbf{I}_C\mathbf{III}_C^{-1/3}$ defines the first invariant of the unimodular right Cauchy–Green tensor $\overline{\boldsymbol{C}} = (\det\boldsymbol{C})^{-1/3}\boldsymbol{C}$ resulting from the multiplicative decomposition of the deformation gradient into a volumetric and an iso-choric part (see [8] and the literature cited therein). The specific SEF has been chosen because it describes a compressible deformation and is implemented in many standard FE codes. Previous studies addressing closed form solutions for compressible materials under finite deformations, see for example [9–11], consider special SEFs different than the common ones in standard FE codes.

The invariants of $\boldsymbol{C}$ and $\boldsymbol{b}$ are equivalent so in the following we mainly use $\boldsymbol{b}$

$$\rho_R\psi(\boldsymbol{b}) = \frac{K}{2}(J-1)^2 + c_{10}(\mathbf{I}_{\bar{b}} - 3) \tag{8}$$
$$= \frac{K}{2}(\mathbf{III}_b^{1/2} - 1)^2 + c_{10}(\mathbf{I}_b\mathbf{III}_b^{-1/3} - 3). \tag{9}$$

For the Neo-Hooke models (8) and (4) are explicitly expressed as:

[1] Abaqus is a trademark of ABAQUS, Inc., Rising Sun Mills, Providence, RI, USA.