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Research Article

Estimation of region of attraction for polynomial nonlinear systems: A numerical method

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ABSTRACT

This paper introduces a numerical method to estimate the region of attraction for polynomial nonlinear systems using sum of squares programming. This method computes a local Lyapunov function and an invariant set around a locally asymptotically stable equilibrium point. The invariant set is an estimation of the region of attraction for the equilibrium point. In order to enlarge the estimation, a subset of the invariant set defined by a *shape factor* is enlarged by solving a sum of squares optimization problem. In this paper, a new algorithm is proposed to select the *shape factor* based on the linearized dynamic model of the system. The shape factor is updated in each iteration using the computed local Lyapunov function from the previous iteration. The efficiency of the proposed method is shown by a few numerical examples.

1. Introduction

Region of attraction (ROA) of a locally asymptotically stable equilibrium point is an invariant set such that all trajectories starting inside this set converge to the equilibrium point. ROA is an important tool in the stability analysis of systems, because the size of the ROA shows that how much the initial points can be far away from the equilibrium point and trajectories can still converge. Finding the exact ROA is in general a very difficult problem. An alternative is to estimate the ROA by computing the largest possible invariant subset of the ROA. In many applications, finding the stable equilibrium points for a nonlinear system is not sufficient to analysis the behavior of system. Because, in practice, a stable equilibrium point with a very small neighborhood may not be so much different comparing to an unstable equilibrium point. Besides, the autonomous nonlinear dynamical system can have several equilibrium points or limit cycles such that the trajectories might converge to each of these points or cycles in case they are stable. Therefore, estimating the stability region of a nonlinear system is a topic of significant importance and has been studied extensively for example in [1–15]. Most computational methods aim to compute the boundary of an invariant set inside the ROA. These methods can be split into Lyapunov and non-Lyapunov methods. Lyapunov methods compute a Lyapunov function (LF) as a local stability certificate and sublevel sets of this LF provide invariant subsets of the ROA. With the recent advances in polynomial optimization based on sum of squares (SOS) relaxations, it is possible to search for polynomial LFs for systems with polynomial and/or rational dynamics. In the literature, the following forms of Lyapunov candidate functions have been employed to estimate the ROA for nonlinear systems: Rational LFs, Polyhedral LFs, Piecewise affine LFs, Polynomial LFs such as Single LFs and Composite polynomial LFs. Rational LFs that approach infinity on the boundary of the ROA are constructed iteratively in [1] motivated by Zubov's work. References [1,2] presented methods based on the concept of a maximal LF, for estimating the ROA of an autonomous nonlinear system. In [3] a nonlinear quadratic system with a locally asymptotically stable equilibrium point in origin was considered. A method is then proposed to determine whether a given polytope belongs to the ROA of the equilibrium using polyhedral LFs. The authors of [4] proposed a method to construct piecewise affine LFs. They suggested a fan-like triangulation around the equilibrium. They showed that if a two dimensional system has an exponentially stable equilibrium, there will be a local triangulation scheme such that a piecewise affine LF exist for the system. The method

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proposed in [5] consists of estimating the ROA via the union of a continuous family of polynomial Lyapunov estimates rather than via one Lyapunov estimate. This method is formulated as a convex Linear Matrix Inequality (LMI) optimization by considering stability conditions for all candidate LFs at the same time [6,16]. In [7], stability analysis and controller synthesis of polynomial systems based on polynomial LFs were investigated. To search for LFs as well as the controllers and to prove local stability, the problem was formulated as iterative SOS optimization problems. In this work, the size of the ROA of the system was also estimated. The proposed method in [7] uses a variable-sized region defined by a *shape factor* to enlarge the estimation of the ROA. The goal is to find the largest sublevel set of a local LF that includes the largest possible shape factor region. Following [7], the authors of [8] proposed using bilinear SOS programming for enlarging a provable ROA of polynomial systems by polynomial LFs. Similar to [7], a polynomial was employed as a *shape factor* to enlarge the ROA estimation. For the same objective, the level sets of a polynomial LF of higher degree were employed because their level sets are richer than that of quadratic LFs. However, the number of optimization decision variables grows extremely fast as the degree of LF and the state dimension increases. In order to keep the number of decision variables low, using pointwise maximum or minimum of a family of polynomial functions was proposed. In [9], a methodology is proposed to generate LF candidates satisfying necessary conditions for bilinear constraints utilizing information from simulations. Qualified candidates were used to compute invariant subsets of the ROA and to initialize various bilinear search strategies for further optimization. In addition to Lyapunov-based methods, there are non-Lyapunov methods such as [10] that focus on topological properties of the ROA. For a survey of results, as well as an extensive set of examples, the reader is referred to [11].

In the last years, due to the importance of estimating the ROA in several fields such as clinical [17,18], economy [19], traffics [20], biological systems [21], chemical processes [22] etc., the ROA estimation has received considerable attention.

This paper is motivated by the work in [7] that uses a *shape factor* to enlarge the estimation of the ROA. It will be shown that the choice of a proper *shape factor* is very important. However, no systematic method has been proposed to select or update the *shape factor*. Therefore, in this paper, we present a general algorithm for using a proper shape factor to enlarge the ROA estimation for nonlinear systems with polynomial vector fields. It will be shown that the proposed method is able to compute an estimation of the ROA of a benchmark problem that, to the best of our knowledge, is larger than the results obtained by existing methods.

This paper is organized as follows: Section 2 contains mathematical preliminaries. Problem statement and a Lyapunov-based method to estimate the ROA for nonlinear systems is explained in Section 3. Then, we propose the main result, an algorithm for selecting a shape factor to improve the estimation of the ROA in Section 4. Some numerical examples and simulation results have been shown in Section 5 to show the efficiency of the proposed algorithm. The paper closes with a conclusion and outlook in Section 6.

2. Mathematical preliminaries

Let the notation be as follows:

$\mathbb{R}$, $\mathbb{Z}^+$: the real number set and the positive integer set.
$\mathbb{R}^n$: an n-dimensional vector space over the field of the real numbers.
$\mathfrak{R}_n$: the set of all polynomials in n variables.

Consider the autonomous nonlinear dynamical system

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector and $x(0) = x_0$ is the initial state at $t = 0$ and $f \in \mathfrak{R}_n$ is a vector polynomial function of x with $f(0) = 0$. The origin is assumed to be locally asymptotically stable.

Definition 1. When the origin is asymptotically stable, the ROA of the origin is defined as

$$\Omega := \{x_0 | \lim_{t\to\infty} \varphi(t, x_0) = 0\} \tag{2}$$

where $\varphi(t, x_0)$ is a solution of Eq. (1) that starts at initial state x_0.

Definition 2. A monomial m_a in n variables is a function defined as

$$m_a := x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \tag{3}$$

for $a_i \in \mathbb{Z}^+$. The degree of m_a is defined as $\deg m_a := \sum_{i=1}^n a_i$.

Definition 3. A polynomial f in n variables is a finite linear combination of monomials,

$$f := \sum_a c_a m_a = \sum_a c_a x^a \tag{4}$$

with $c_a \in \mathbb{R}$. The degree of f is defined as $\deg f := \max_a \deg m_a$ (c_a is non-zero).

Definition 4. Define $\sum_n$ to be the set of SOS polynomials in n variables.

$$\sum_n := \{p \in \mathfrak{R}_n | p = \sum_{i=1}^k f_i^2, \; f_i \in \mathfrak{R}_n, i = 1, \dots, k.\} \tag{5}$$

Obviously if $p \in \sum_n$, then $p(x) \geq 0 \quad \forall x \in \mathbb{R}^n$. A polynomial, $p \in \sum_n$ if $\exists 0 \leq Q \in \mathbb{R}^{r\times r}$ such that

$$p(x) = z^T(x) Q z(x) \tag{6}$$

with $z(x)$ a vector of suitable monomials [7].

Definition 5. Given $\{p_i\}_{i=0}^m \in \mathfrak{R}_n$, generalized S-procedure states: if there exist $\{s_i\}_{i=1}^m \in \sum_n$ such that $p_0 - \sum_{i=1}^m s_i p_i \in \sum_n$, then [23]

$$\cap_{i=1}^m \{x \in \mathbb{R}^n | p_i \geq 0\} \subseteq \{x \in \mathbb{R}^n | p_0 \geq 0\} \tag{7}$$

3. Estimating the region of attraction

In general, exact computation of the ROA is a difficult task [9]. Hence, one should look for a numerical method to find the best possible estimation of the ROA. Since the Lyapunov technique is a powerful method in investigating the stability of nonlinear systems [24,25], in this section, a Lyapunov-based method is described to estimate the ROA by a LF sublevel set. The numerical algorithm in this section is based on a lemma from [26], that will be described in the following.

If for an open connected set S in $\mathbb{R}^n$ containing 0, there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$ and the following conditions hold

$$V(0) > 0, \;\; \dot{V}(x) < 0, \;\; \forall x \neq 0 \;\; in \;\; S \tag{8}$$

then every invariant set contained in S is also contained in the ROA of equilibrium point, but S itself need not be contained in ROA of 0. For finding such invariant sets, an easy way is to use so-called level sets of the (local) LF V. Let c be a positive value, and consider the set

$$M_V(c) = \{x \in \mathbb{R}^n | V(x) \leq c\} \tag{9}$$

Now, the connected level set $M_V(c)$ containing 0, is a subset of the ROA of 0 [26].