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Research Article

Finite-time stabilization for a class of stochastic nonlinear systems via output feedback

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ABSTRACT

This paper investigates the problem of global finite-time stabilization in probability for a class of stochastic nonlinear systems. The drift and diffusion terms satisfy lower-triangular or upper-triangular homogeneous growth conditions. By adding one power integrator technique, an output feedback controller is first designed for the nominal system without perturbing nonlinearities. Based on homogeneous domination approach and stochastic finite-time stability theorem, it is proved that the solution of the closed-loop system will converge to the origin in finite time and stay at the origin thereafter with probability one. Two simulation examples are presented to illustrate the effectiveness of the proposed design procedure.

1. Introduction

In this paper, we consider the problem of finite-time stabilization via output feedback for a class of stochastic nonlinear systems described by

$$
\begin{aligned}
dx_1(t) &= x_2(t)\,dt + f_1(x(t),u(t))\,dt + g_1^T(x(t),u(t))\,d\omega(t),\\
dx_2(t) &= x_3(t)\,dt + f_2(x(t),u(t))\,dt + g_2^T(x(t),u(t))\,d\omega(t),\\
&\vdots\\
dx_n(t) &= u(t)\,dt + f_n(x(t),u(t))\,dt + g_n^T(x(t),u(t))\,d\omega(t),\\
y(t) &= x_1(t)
\end{aligned}
\tag{1}
$$

where $x(t) = (x_1(t), \ldots, x_n(t))^T \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$ are the system states, control input and output, respectively. $\omega(t)$ is an r-dimensional standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with Ω being a sample space, $\mathcal{F}$ being a σ-field, $\mathcal{F}_t$ being a filtration and P being a probability measure. The drift terms $f_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and the diffusion terms $g_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^r$, $i = 1, \ldots, n$, are Borel measurable, continuous in system states and satisfy $f_i(0,0) = 0$ and $g_i(0,0) = 0$.

In the nonlinear control community, finite-time stabilization is one of the most fundamental and challenging problems. In contrast to the commonly used notion of asymptotic stability, finite-time stability requires essentially that a control system should be stable in the sense of Lyapunov and its trajectories tend to zero in finite time. It was demonstrated in [1] that finite-time stable systems might have not only faster convergence but also better robustness and disturbance rejection properties. The work [2] provided a solid foundation for finite-time stability theory of continuous autonomous systems, which gave a judging criterion on finite-time stability. Then, some conditions for finite-time stability have been presented for continuous systems [3] and non-autonomous continuous systems [4]. In the literature, several results on finite-time state feedback stabilization have been achieved in [5–7] and the references therein. However, since finite-time stabilizers are generally not smooth, their design methods are sophisticated, especially when some states are not measurable. Based on a "finite-time separation principle", global finite-time stabilization via output feedback can be achieved for the double integrator system in [8]. The work [9] has developed a novel systematic design method, namely homogeneous domination approach, which provides us a new perspective to deal with the output feedback control problem for nonlinear systems and leads to several stabilization results [10–12]. By coupling the homogeneous domination approach and finite-time stabilization technique, an output feedback controller was constructed to global stabilize a class of lower-triangular nonlinear systems [13] and upper-triangular ones [14]. Moreover, with unknown output gain, the problem of global finite-time stabilization has been addressed in [15,16].

In spite of these developments, the above-mentioned results cannot be generalized easily to a class of stochastic nonlinear systems. It is well known that stochastic modeling has come to play an important role in many branches of science and industry.

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Florchinger extended the concept of control Lyapunov functions and Sontag's stabilization formula to stochastic setting in [17], which leads to more stabilization results [18–23] and the references therein. However, each of them described the asymptotic behavior of trajectories for a class of stochastic nonlinear systems as time tends to infinity. Recently, the work [24] has presented the concept of finite-time stability in probability for stochastic systems and has proved the stochastic finite-time stability theorem. Subsequently, for a class of stochastic nonlinear systems in strict-feedback form, the work [25] designed a continuous state-feedback controller to guarantee the global finite-time stability in probability and our recent work [26] solved the finite-time stabilization problem by dynamic state-feedback. However, only *state feedback* was considered, which requires all the system states to be measurable. Immediately, one may ask the following interesting questions: *Is it possible to relax the growth conditions for nonlinear functions? Under these weaker conditions, how can one design an output feedback controller to make* (1) *globally finite-time stable in probability*?

Motivated by the design of finite-time stabilizer in deterministic cases [13,14], and stochastic finite-time stability theorem proposed in [24], we aim to solve the problem of global finite-time stabilization for a class of stochastic nonlinear systems via *output feedback*. In order to settle this problem, we first design a homogeneous output feedback controller for the nominal system. Then, a scaling gain is introduced to the controller to dominate the perturbing nonlinearities. By appropriately choosing the scaling gain, the closed-loop system can be rendered globally finite-time stable in probability. Furthermore, we extend the result to a class of upper-triangular stochastic nonlinear systems. The main contributions of this paper are as follows:

(i) Compared with deterministic cases [13,14], this paper extends the global finite-time stabilization results to a class of stochastic nonlinear systems according to stochastic finite-time stability theorem.
(ii) The uncertain nonlinearities are functions of both measurable and unmeasurable states. Based on the homogeneous observer construction, an output feedback controller guarantees the closed-loop system finite-time stable in probability.

Notations: $\mathbb{R}_+$ denotes the set of all nonnegative real numbers, and $\mathbb{R}^n$ denotes the real n-dimensional space. $\mathbb{R}^+_{odd}=:\{q\in\mathbb{R}: q\geq 0$ is a ratio of two odd integers$\}$. For a given vector or matrix X, X^T represents its transpose; $\mathrm{Tr}\{X\}$ represents its trace when X is square; $\|\cdot\|$ denotes the Euclidean norm of a vector X or the Frobenius norm of a matrix X. $\mathcal{C}^i$ denotes the set of all functions with continuous ith partial derivatives; $\mathcal{K}$ denotes the set of all functions, $\mathbb{R}_+\to\mathbb{R}_+$, which are continuous, strictly increasing and vanishing at zero; $\mathcal{K}_\infty$ denotes the set of all functions which are of class $\mathcal{K}$ and unbounded; $a\wedge b$ means the minimum of a and b.

2. Preliminary results

In this section, we present some useful definitions and lemmas which play very important roles in this paper. Consider the following stochastic nonlinear system:

$$dx(t)=f(x(t))\,dt+g^T(x(t))\,d\omega(t),\quad x(0)=x_0\in\mathbb{R}^n \tag{2}$$

where $x(t)\in\mathbb{R}^n$ is the system state and $\omega(t)$ is an r-dimensional standard Wiener process defined on a probability space $(\Omega,\mathrm{F},\mathrm{F}_t,P)$. The Borel measurable functions $f:\mathbb{R}^n\to\mathbb{R}^n$ and $g^T:\mathbb{R}^n\to\mathbb{R}^{n\times r}$ are continuous in x that satisfy $f(0)=0$ and $g(0)=0$.

Lemma 2.1 (*Skorokhod [27]*). *Suppose that $f(x(t))$ and $g(x(t))$ are continuous with respect to their variables and satisfy the linear growth condition:*

$$\|f(x(t))\|^2+\|g(x(t))\|^2\leq K(1+\|x(t)\|^2) \tag{3}$$

for $K>0$. Then given any x_0 independent of $\omega(t)$, (2) has a continuous solution with probability one.

Definition 2.1 (*Khoo et al. [25]*). The trivial solution of (2) is said to be finite-time stable in probability if the solution exists for any initial value $x_0\in\mathbb{R}^n$, denoted by $x(t;x_0)$. Moreover, the following statements hold:

(i) *Finite-time attractiveness in probability*: For every initial value $x_0\in\mathbb{R}^n\backslash\{0\}$, the first hitting time $\tau_{x_0}=\inf\{t;x(t;x_0)=0\}$, which is called the stochastic settling time, is finite almost surely, that is, $P\{\tau_{x_0}<\infty\}=1$.
(ii) *Stability in probability*: For every pair of $\varepsilon\in(0,1)$ and $r>0$, there exists a $\delta=\delta(\varepsilon,r)>0$ such that $P\{\|x(t;x_0)\|<r,\ \forall t\geq 0\}\geq 1-\varepsilon$, whenever $\|x_0\|<\delta$.
(iii) The solution $x((t+\tau_{x_0});x_0)$ is unique for $t\geq 0$.

Definition 2.2 (*Florchinger [17]*). For any given $V(x(t))\in\mathcal{C}^2$ associated with stochastic system (2), the infinitesimal generator $\mathcal{L}$ is defined as $\mathcal{L}V(x)=(\partial V/\partial x)f(x)+\frac{1}{2}\,\mathrm{Tr}\{g(x)(\partial^2V/\partial x^2)g^T(x)\}$, where $\frac{1}{2}\,\mathrm{Tr}\{g(x)(\partial^2V/\partial x^2)g^T(x)\}$ is called as the Hessian term of $\mathcal{L}$.

Lemma 2.2 (*Khoo et al. [25]*). *For system (2), if there exist a Lyapunov function $V:\mathbb{R}^n\to\mathbb{R}_+$, $\mathcal{K}_\infty$ class functions μ_1 and μ_2, positive real numbers $c>0$ and $0<\gamma<1$, such that for all $x\in\mathbb{R}^n$ and $t\geq 0$,*

$$\mu_1(\|x\|)\leq V(x)\leq\mu_2(\|x\|), \tag{4}$$

$$\mathcal{L}V(x)\leq -c\cdot(V(x))^\gamma \tag{5}$$

then the trivial solution of (2) is finite-time attractive and stable in probability.

Definition 2.3 (*Kawski [28]*). For real numbers $r_i>0$, $i=1,\dots,n$, and fixed coordinates $(x_1,\dots,x_n)\in\mathbb{R}^n$, $\forall\varepsilon>0$.

- the dilation $\Delta_\varepsilon(x)$ is defined by $\Delta_\varepsilon(x)=(\varepsilon^{r_1}x_1,\dots,\varepsilon^{r_n}x_n)$, $\forall\varepsilon>0$, with r_i being called as the weights of the coordinates. For simplicity of notation, we define dilation weight $\Delta=(r_1,\dots,r_n)$.
- a function $V\in\mathcal{C}(\mathbb{R}^n,\mathbb{R})$ is said to be homogeneous of degree τ if there is a real number $\tau\geq 0$ such that $\forall x\in\mathbb{R}^n\backslash\{0\}$, $V(\Delta_\varepsilon(x))=\varepsilon^\tau V(x_1,\dots,x_n)$.
- a vector field $f\in\mathcal{C}(\mathbb{R}^n,\mathbb{R}^n)$ is said to be homogeneous of degree τ if there is a real number $\tau\geq -\min_{1\leq i\leq n}\{r_i\}$ such that for $i=1,\dots,n$, $\forall x\in\mathbb{R}^n\backslash\{0\}$, $f_i(\Delta_\varepsilon(x))=\varepsilon^{\tau+r_i}f_i(x_1,\dots,x_n)$.
- a homogeneous p-norm is defined as $\|x\|_{\Delta,p}=(\sum_{i=1}^n|x_i|^{p/r_i})^{1/p}$, $\forall x\in\mathbb{R}^n$, for a constant $p\geq 1$. For simplicity, we choose $p=2$ and write $\|x\|_\Delta$ for $\|x\|_{\Delta,2}$.

Lemma 2.3. *Suppose c and d are two positive real numbers. Given any positive number $\gamma>0$, the following inequality holds:*

$$|x|^c|y|^d\leq\frac{c}{c+d}\gamma|x|^{c+d}+\frac{d}{c+d}\gamma^{-\frac{c}{d}}|y|^{c+d}.$$

Lemma 2.4. *For $x\in\mathbb{R}$, $y\in\mathbb{R}$, and $p\geq 1$, the following inequalities hold:*

$$|x+y|^p\leq 2^{p-1}|x^p+y^p|,$$

$$(|x|+|y|)^{1/p}\leq|x|^{1/p}+|y|^{1/p}\leq 2^{(p-1)/p}(|x|+|y|)^{1/p}.$$