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Research Article

Robust stability and stabilization of fractional order linear systems with positive real uncertainty

Yingdong Ma, Junguo Lu *, Weidong Chen

Department of Automation, Shanghai Jiao Tong University, and Key Laboratory of System Control and Information Processing, Ministry of Education of China, No. 800 Dong Chuan Road, Min Hang, Shanghai 200240, P.R. China

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ABSTRACT

This paper investigates the robust stability and stabilization of fractional order linear systems with positive real uncertainty. Firstly, sufficient conditions for the asymptotical stability of such uncertain fractional order systems are presented. Secondly, the existence conditions and design methods of the state feedback controller, static output feedback controller and observer-based controller for asymptotically stabilizing such uncertain fractional order systems are derived. The results are obtained in terms of linear matrix inequalities. Finally, some numerical examples are given to validate the proposed theoretical results.

1. Introduction

Fractional order control systems have attracted growing attention and interest of physicists and engineers from an application point of view recently (see [1–7, and the references therein]). On the one hand, this is mainly due to the fact that many real-world physical systems in interdisciplinary fields can be well characterized by fractional order differential equations involving the so-called fractional derivatives and integrals (for an introduction to this theory, see [1]). In particular, it has been shown that viscoelastic materials having memory and hereditary effects [8], biomedical systems [9–11], dynamical processes such as semi-infinite lossy RC transmission [12], mass diffusion and heat conduction [13] can be more adequately modeled by fractional order models than the traditional integer order models. In addition, fractional order derivatives and integrals also provide a powerful instrument for modeling dynamical processes in fractal media [14]. This is a significant advantage of the fractional order models in comparison with integer order models, where such effects or geometry have been neglected [14]. On the other hand, with the success in the synthesis of real noninteger differentiator and the emergence of new electrical circuit element called "fractance" [15,16], fractional order controllers [14,17–22] have been designed and applied to control a variety of dynamical processes, including integer order and fractional order systems, so as to enhance the robustness and performance of the control systems.

Stability and stabilization is fundamental to fractional order control systems [23–26]. In practice, there exist some uncertainties in the model due, for example, to some uncertain physical parameters, parametrical variations in time, neglected dynamics and so on. These uncertainties, which have to be considered for modeling and analyzing the system, can be introduced through various forms. There have been some stability results about the fractional order systems with interval uncertainties [27–32], polytopic uncertainties [33,34], or norm-bounded uncertainties [35]. For example, the robust stability problem of fractional order linear time-invariant (FO-LTI) interval systems described in the transfer function form was investigated in [30]. The robust stability problem of FO-LTI interval systems described in the state-space form was first considered in [29] by using the matrix perturbation theory. In [28], based on Lyapunov inequality, a new robust stability checking method was proposed for FO-LTI interval uncertain systems. However the results in [28,29] only are sufficient conditions. In [27], the necessary and sufficient condition for the robust stability of FO-LTI interval systems with fractional orders α, $1 \le \alpha < 2$, was presented. In [31,32], robust stability and stabilization problems of FO-LTI interval systems were investigated by using the linear matrix inequality method. In [36,37], sufficient conditions for the robust stability and stabilization of a class of FO-LTI interval systems with linear coupling relationships among the fractional order, the system

* Corresponding author. Tel.: +86 21 34205004; fax: +86 21 34204302.
E-mail addresses: jglu@sjtu.edu.cn, junguolu@hotmail.com (J. Lu).

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matrix and the input matrix were derived. In [35], observer-based controller and static output feedback controller for uncertain fractional order systems with norm-bounded uncertainty via linear matrix inequality approach were designed. In [38], synchronization of uncertain chaotic fractional order Duffing–Holmes systems was achieved by using the sliding mode control. In [39], an adaptive fuzzy sliding mode control for synchronizing two different uncertain fractional order time-delay chaotic systems was investigated. In [40], the sliding mode controller for an uncertain chaotic fractional order economic system was designed.

It is well known that the interval uncertainty description, polytopic uncertainty description, and norm-bounded uncertainty description only can capture gain uncertainty [41,42]. When uncertainty phase information is available, these uncertainty description may lead to conservative results [41,42]. A way for accounting phase information is to apply the positivity theorem and more precisely to model the uncertainty through a positive real uncertainty matrix as in [41–44]. Note that positive real uncertainty exists in many real systems, and the robust stability and stabilization problem of integer order systems with positive real uncertainty has been studied in [41–44]. To the best of our knowledge, there are few results concerning robust stability and stabilization of fractional order linear systems with positive real uncertainty. Moreover, in most practical applications, the system state vector is not always accessible and only the partial information is available via measured output. In this case, the output feedback control or observer-based control is often needed.

With the above motivation, the robust stability and stabilization of fractional order linear systems with positive real uncertainty will be investigated. Firstly, sufficient conditions for the asymptotical stability of such uncertain fractional order systems are presented. Secondly, the existence conditions and design methods of the state feedback controller, static output feedback controller and observer-based controller for asymptotically stabilizing such uncertain fractional order systems are derived. The results are obtained in terms of linear matrix inequalities. Finally, some numerical examples are given to validate the proposed theoretical results.

The rest of this paper is organized as follows: in Section 2, the problem formulation and some necessary preliminaries are presented. In Section 3, robust stability conditions of uncertain fractional order systems with positive real uncertainty are derived. In Section 4, robust stabilizable conditions of such uncertain fractional order systems via state feedback control, static output feedback control and observer-based output feedback control and the design methods of the corresponding controllers are derived. For illustration of the effectiveness of the proposed theoretical results, numerical examples are presented in Section 5. Finally, some conclusions are given in Section 6.

Notations: We denote by M^T the transpose of M, by $\overline{M}$ the conjugate of M, by M^* the transpose conjugate of M, by $\bar{z}$ the conjugate of the scalar number z, by $\mathrm{Re}(z)$ its real part and by $\mathrm{Im}(z)$ its imaginary part. I_n is the identity matrix of order n. Matrices, if not explicitly stated, are assumed to have appropriate dimensions. $\otimes$ is the Kronecker product of two matrices and $(A\otimes B)(C\otimes D)=(AC)\otimes(BD)$. $\mathbf{i}$ denotes the imaginary unit. $Sym\{X\}$ denotes the expression X^*+X. The notation $\bullet$ stands for the symmetric component in matrix.

2. Problem formulation and preliminaries

Consider the following uncertain fractional order linear system:

$$\begin{cases} \dfrac{d^\alpha x(t)}{dt^\alpha}=(A+\Delta A)x(t)+(B+\Delta B)u(t), \\ y(t)=Cx(t), \end{cases} \tag{1}$$

where α is the fractional commensurate order, $x(t)\in\mathbb{R}^n$ denotes the state vector, $u(t)\in\mathbb{R}^l$ is the control input, $y(t)\in\mathbb{R}^m$ is the output vector, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times l}$ and $C\in\mathbb{R}^{m\times n}$ are constant matrices. ΔA and ΔB are time-invariant matrices with parametric uncertainties, and are assumed to be of the form (see for example [42–44])

$$[\Delta A \ \ \Delta B]=M\Delta(\zeta)[N_1 \ \ N_2], \tag{2}$$

$$\Delta(\zeta)=F(\zeta)[I+JF(\zeta)]^{-1}, \tag{3}$$

$$Sym\{J\}>0, \tag{4}$$

where $M\in\mathbb{R}^{n\times m_0}$, $N_1\in\mathbb{R}^{m_0\times n}$, $N_2\in\mathbb{R}^{m_0\times l}$, and $J\in\mathbb{R}^{m_0\times m_0}$ are known real constant matrices. The uncertain matrix $F(\zeta)\in\mathbb{R}^{m_0\times m_0}$ satisfies

$$Sym\{F(\zeta)\}\ge 0, \tag{5}$$

where $\zeta\in\Omega$ with Ω being a compact set.

Remark 1. It can be verified that the condition (4) guarantees that $I-JF(\zeta)$ is invertible for all $F(\zeta)$ satisfying (5). Therefore, $\Delta(\zeta)$ in (3) is well defined [42–44].

In this paper, the following Caputo definition is adopted for fractional derivatives of order α of function $f(t)$, since the Laplace transform of the Caputo derivative allows utilization of initial values of classical integer-order derivatives with clear physical interpretations [1]:

$$D^\alpha f(t)=\frac{d^\alpha f(t)}{dt^\alpha}=\frac{1}{\Gamma(\alpha-m)}\int_0^t\frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}, \tag{6}$$

where m is an integer satisfying $m-1<\alpha\le m$ and $\Gamma(\cdot)$ is the well-known Euler Gamma function $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt$. From a mathematical point of view, the fractional order can be any real even complex number. In engineering applications, α often lies in (0, 2), and is a real number related to physical parameters. Therefore, this paper focuses on the robust stability and stabilization problem of uncertain fractional order systems (1) where α is a real number in (0, 2).

To proceed, we need the following assumption and lemmas.

Lemma 1 (*Matignon [25], Sabatier et al. [45]*). *Let $A\in\mathbb{R}^{n\times n}$ and $0<\alpha<2$. Then, a necessary and sufficient condition for the asymptotical stability of $d^\alpha x(t)/dt^\alpha=Ax(t)$ is*

$$|\arg(spec(A))|>\alpha\tfrac{\pi}{2},$$

where $spec(A)$ is the spectrum of all eigenvalues of A.

Lemma 2 (*Farges et al. [33]*). *Let $A\in\mathbb{R}^{n\times n}$, $0<\alpha<1$ and $\theta=(1-\alpha)\pi/2$. The fractional order system $d^\alpha x(t)/dt^\alpha=Ax(t)$ is asymptotically stable if and only if there exists a positive definite Hermitian matrix $X=X^*>0$, $X\in\mathbb{C}^{n\times n}$ such that*

$$(rX+\bar{r}\overline{X})^TA^T+A(rX+\bar{r}\overline{X})<0, \tag{7}$$

where $r=e^{\theta\mathbf{i}}$.

Lemma 3 (*Sabatier et al. [45]*). *Let $A\in\mathbb{R}^{n\times n}$, $1\le\alpha<2$ and $\theta=\pi-\alpha\pi/2$. The fractional order system $d^\alpha x(t)/dt^\alpha=Ax(t)$ is asymptotically stable if and only if there exists a positive definite matrix $X=X^T>0$, $X\in\mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix}(A^TX+XA)\sin\theta & (XA-A^TX)\cos\theta\\ \bullet & (A^TX+XA)\sin\theta\end{bmatrix}<0. \tag{8}$$

Remark 2. Defining

$$\Theta=\begin{bmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}, \tag{9}$$