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Robust stabilization of uncertain nonlinear slowly-varying systems: Application in a time-varying inertia pendulum

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ABSTRACT

This paper considers the problem of robust stabilization of nonlinear slowly-varying systems, in the presence of model uncertainties and external disturbances. The main contribution of this paper is an extension of the Slowly-Varying Control Lyapunov Function (SVCLF) technique to design a robust stabilizing controller for nonlinear slowly-varying systems with matched uncertainties. In the proposed strategy, the Lyapunov redesign method is utilized to design a robust control law. This method, originally, leads to a discontinuous controller which suffers from chattering. In this paper, this problem is removed by using a saturation function with a high slope, as an approximation of the signum function. Since, using the saturation function leads to loss of asymptotic stability and, instead, guarantees only the boundedness of the system's states; therefore, some sufficient conditions are proposed to guarantee the asymptotic stability of the closed-loop uncertain nonlinear slowly-varying system (without chattering). Also, in order to show the applicability of the proposed method, it is applied to a time-varying inertia pendulum. The efficiency of the designed controller is demonstrated through analysis and simulations.

1. Introduction

Among the nonlinear time-varying systems, an important category is the nonlinear systems with slowly-varying parameters. Such systems are called slowly-varying systems [1] and may be considered between time-invariant and time-varying systems. Slowly-varying systems have many applications in physics and control engineering [2–5]. For these systems, the algorithms developed for time-invariant systems, may cause instability. On the other hand, the general time-varying based methods may be too conservative and complicated due to control law.

A considerable amount of scientific works have been done in the area of analysis of nonlinear slowly-varying systems, for instance see [6–8]. However, there are a few works presenting a framework to design a nonlinear stabilizing controller for nonlinear slowly-varying systems and most of the existing papers in this field, are concentrated on linear slowly-varying systems [9–11]. Authors of [12–14] have proposed stabilizing control laws for nonlinear slowly-varying systems, based on the *Control Lyapunov Function* (*CLF*) method. The CLF method has been, originally, introduced by Sontag to stabilize the nonlinear time-invariant systems [15]. The CLF-based stabilization method for nonlinear time-varying systems leads to a controller with a complicated time-varying structure [16]. The proposed controllers in [13,14], which were called SVCLF controller, are structurally as simple as the CLF controller for nonlinear time-invariant systems. In [13] the SVCLF controller has been designed for nonlinear slowly-varying systems with a scalar input and a scalar slowly-varying parameter while the authors of [14] have extended this method to nonlinear multi input time-varying systems with slowly-varying parameters However, in spite of the advantages of the proposed methods in [12–14], model uncertainties and external disturbances, which exist in many of practical systems [17–20], have not been considered in the model of the nonlinear slowly-varying systems.

The main result of this paper is an extension of the SVCLF technique to design a robust stabilizing controller for uncertain nonlinear slowly-varying systems with matched uncertainties. The proposed strategy utilizes the Lyapunov redesign method to conquer the uncertainties. In the design procedure, first, the nominal controller is designed for the nominal system (the slowly-varying system without any uncertainties and external disturbances), and then, using the Lyapunov redesign method, an additional term is added to the control law to guarantee the robust asymptotic stabilization in the presence of uncertainties and external disturbances. The Lyapunov redesign method, originally, leads to a discontinuous controller with *signum* function. Since, discontinuous controllers suffer from chattering, to alleviate this problem; the *signum* function is replaced with the *saturation*

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function with a high slope. However, this causes to loss of the asymptotic stability and, instead, only the boundedness of the system's states could be guaranteed. In this paper, in addition to designing a robust controller for nonlinear slowly-varying systems, some sufficient conditions are also presented which guarantee the asymptotic stability of the closed-loop system (without chattering). In order to show the applicability of the proposed method, it is applied to the time-varying inertia pendulum, which is one of the famous benchmarks among the nonlinear time-varying systems. Computer simulations show the efficiency of the proposed method in robust asymptotic stabilization of the nonlinear time-varying inertia pendulum in the presence of model uncertainties.

2. Problem formulation and preliminaries

Consider the following uncertain nonlinear slowly-varying system:

$$\dot{x} = f(x, \theta(t)) + g(x, \theta(t))[u + d(x, u, \theta(t))] \tag{1}$$

where $x \in D \subset \Re^n$ (D contains the origin) and $u \in \Re^p$ are the state and input vectors, respectively. $\theta(t) \in \Omega \subset \Re^q$ is a vector of time-varying parameters, whose variations are bounded and small enough (the slowly-varying parameters) and the perturbation term $d(x, u, \theta(t))$ is resulted from modeling error or uncertainties and external disturbances, which exists in any practical problem. In a typical situation, $d(x, u, \theta(t))$ is unknown, but some information about it, like an upper bound on $\|d(x, u, \theta(t))\|$ is usually known. In this paper it is assumed that

$$\|d(x, \theta(t))\|_\infty \le \rho(x), \quad \forall (x, \theta) \in D \times \Omega \tag{2}$$

where $\rho(x)$ is a known positive function.

The task is to design a robust controller such that the closed-loop system (1) is asymptotically stable in the presence of $d(x, u, \theta(t))$. Additionally, in order to make the performance quantitative, it is desirable to minimize the following cost function:

$$J = \int_0^\infty (q(x(\tau), \theta(\tau)) + u(\tau)^T u(\tau)) d\tau \tag{3}$$

where $q(x, \theta)$ is a positive-definite function. Since, choosing the cost function is effective on the characteristics of the transient responses of system's states, $q(x, \theta)$ should be selected such that the system's performance to be acceptable in terms of settling time, overshoot and etc.

In the following theorem, the nominal controller is designed (according to SVCLF method) to asymptotically stabilize the following nonlinear slowly-varying system (called the nominal system):

$$\dot{x} = f(x, \theta(t)) + g(x, \theta(t))u \tag{4}$$

For this purpose, considering the parametric Lyapunov function $V(x, \theta(t)) : D \times \Omega \to R$ (which is called the slowly-varying control Lyapunov function), the functions $a(x, \theta(t))$ and $b(x, \theta(t))$ are defined as follows (note that $V_x = \partial V / \partial x$ is assumed as a row vector):

$$a(x, \theta(t)) = V_x f(x, \theta(t)), \tag{5}$$

$$b(x, \theta(t)) = V_x g(x, \theta(t)). \tag{6}$$

Theorem 1. *Considering the nonlinear slowly-varying system (4), suppose that $\gamma(\|x\|)$ is a class K function and there exists a parametric Lyapunov function $V(x, \theta(t)) : D \times \Omega \to R$ such that*

$$\alpha_1(\|x\|) \le V(x, \theta(t)) \le \alpha_2(\|x\|), \qquad \forall (x, \theta(t)) \in D \times \Omega \tag{7}$$

$$a(x, \theta(t)) \le -\gamma(\|x\|), \qquad \forall (x, \theta(t)) \in D \times \Omega \quad \text{where} \quad b(x, \theta(t)) = 0 \tag{8}$$

$$\left\| \frac{\partial V}{\partial \theta} \right\| < \infty, \qquad \forall (x, \theta(t)) \in D \times \Omega \tag{9}$$

where $\alpha_1(.)$ and $\alpha_2(.)$ are class K functions. If $\|\dot{\theta}(t)\|$ (for every $t \in [0, \infty)$) satisfies the condition (10), then the SVCLF controller (11) leads to $\dot{V} \le -\alpha\gamma(\|x\|)$ in the trajectories of the closed-loop system (where $\alpha \in (0, 1)$ is chosen such that the infimum in (10) is positive and $q(x, \theta)$ is given in the cost function (3)).

$$\sup_t \|\dot{\theta}\| \le \inf_{\theta \in \Omega, x \in D} \frac{\sqrt{a^2(x, \theta) + q(x, \theta) b(x, \theta) b^T(x, \theta)} - \alpha\gamma(\|x\|)}{\|\partial V / \partial \theta\|} \tag{10}$$

$$u = k(x, \theta(t)) = \begin{cases} -\dfrac{b^T(x, \theta(t))\left(a(x, \theta(t)) + \sqrt{a^2(x, \theta(t)) + q(x, \theta(t)) b(x, \theta(t)) b^T(x, \theta(t))}\right)}{b(x, \theta(t)) b^T(x, \theta(t))}, & \text{where} \quad b \ne 0 \\ 0, & \text{where} \quad b = 0 \end{cases} \tag{11}$$

Therefore, the control law (11) guarantees the asymptotic stabilization of the nominal system (4).

Proof. See [14]. □

Remark 1. One of the important issues in the design the proposed control law is its link with optimality. Consider the nonlinear system (4) with the cost function (3). Finding the optimal controller for this system leads to solving the following time-varying partial differential equation (which is called time-varying Hamilton–Jacobi–Bellman (HJB) equation):

$$V_t^* + q + V_x^* f - \frac{1}{4} V_x^* g g^T V_x^{*T} = 0$$

The optimizing control action is $u^* = -0.5 g^T V_x^{*T}$ which may be evaluated after solving HJB equation and finding V_x^* [21]. Unfortunately, the HJB equation is not analytically solvable for most practical systems. Therefore, approximate techniques have been used to solve this equation and estimate V_x^* which leads to suboptimal controllers. It has been shown in [14] that the SVCLF controller is also a suboptimal controller and the response of the system (4) with this controller may be very close to its optimal solution.

3. Robust stabilization of nonlinear slowly-varying systems

In this section, an additional term (v) is designed (based on the Lyapunov redesign method), such that the overall feedback law ($u = k(x, \theta(t)) + v$) guarantees robust asymptotic stabilization of the nonlinear system (1) in the presence of the unknown vector function $d(x, u, \theta(t))$.

Theorem 2. *Consider the nonlinear slowly-varying system (1). Take $r > 0$, such that $B_r = \{x \in R^n : \|x\| \le r\} \subset D$ and suppose that the elements of v are as follows:*

$$v_i = \begin{cases} -\eta(x)\mathrm{sgn}(b_i), & \eta(x)|b_i| \ge \varepsilon \\ -\eta^2(x)\frac{b_i}{\varepsilon}, & \eta(x)|b_i| < \varepsilon \end{cases} \quad \text{for} \quad i = 1, \cdots, p \tag{12}$$

where b_i is the i$^{\text{th}}$ element of the row vector $b(x, \theta(t))$, defined in (6), and $\eta(x) \ge \rho(x)$ is a positive continuous function of the state variables. Also, $\varepsilon < 4(1-\beta)\alpha\gamma[\alpha_2^{-1}(\alpha_1(r))]/p$ where β is an arbitrary value belonging to $(0, 1)$. Also, $\alpha_1(.)$, $\alpha_1(.)$, $\gamma(.)$ and α have been introduced in Theorem 1. The feedback law $u = k(x, \theta) + v$ (where $k(x, \theta)$ is the nominal controller (11)) guarantees that for any $\|x(t_0)\| < \alpha_2^{-1}(\alpha_1(r))$, there exists a finite time T such that

$$\|x(t)\| \le \alpha_1^{-1}(\alpha_2(\mu)), \quad \forall t \ge t_0 + T \tag{13}$$

where $\mu = \gamma^{-1}(p\varepsilon / [4(1-\beta)\alpha])$.

Proof. According to the result of Theorem 1, there exists a Lyapunov function $V(x, \theta(t))$, for the nominal system (4), such that

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}[f(x, \theta(t)) + g(x, \theta(t))k(x, \theta(t))] \le -\alpha\gamma(\|x\|),$$