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Research Article

A new and simple method to construct root locus of general fractional-order systems

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ABSTRACT

Recently fractional-order (FO) differential equations are widely used in the areas of modeling and control. They are multivalued in nature hence their stability is defined using Riemann surfaces. The stability analysis of FO linear systems using the technique of Root Locus is the main focus of this paper. Procedure to plot root locus of FO systems in s-plane has been proposed by many authors, which are complicated, and analysis using these methods is also difficult and incomplete. In this paper, we have proposed a simple method of plotting root locus of FO systems. In the proposed method, the FO system is transformed into its integer-order counterpart and then root locus of this transformed system is plotted. It is shown with the help of examples that the root locus of this transformed system (which is obviously very easy to plot) has exactly the same shape and structure as the root locus of the original FO system. So stability of the FO system can be directly deduced and analyzed from the root locus of the transformed IO system. This proposed procedure of developing and analyzing the root locus of FO systems is much easier and straightforward than the existing methods suggested in the literature. This root locus plot is used to comment about the stability of FO system. It also gives the range for the amplifier gain k required to maintain this stability. The reliability of the method is verified with analytical calculations.

1. Introduction

Recent years have seen a tremendous upsurge in research activities in the area related to the use of fractional-order (FO) differential equations in modeling and control. FO differential equations are found to provide a more realistic, faithful, and compact representations of many real world, natural and man-made systems. Further, FO controllers, on the other hand, have been able to achieve a better closed-loop performance and robustness, than their integer-order counterparts. An interesting study of fractional-order differential systems appeared in [1] using a stochastic framework. Latest and very exhaustive literature survey about the FC and FO systems is given in [2]. Applications such as viscoelasticity, cell diffusion, biology, signal processing, modeling and control can be found in [3–7]. Various methods of discretization of fractional-order differentiators and integrators are discussed in [8].

Stability is the main concern in control theory. Recently, there has been substantial development in control theory for stability analysis of FO systems. Stability of linear FO systems is discussed in [9–12]. Transfer functions of FO systems consist of polynomials that are not integer-order (IO) but are pseudo-polynomials with non-integer order. Hence it is difficult to evaluate the stability by simply examining its characteristic equation or by finding its roots [13]. The algebraic methods like Routh's criteria cannot be used for polynomials with fractional powers of complex variables. Stability by Routh–Hurwitz criterion for FO systems is presented in [14], but it is a very complicated algorithm. Geometric methods such as Nyquist criteria can be used for the stability check of BIBO systems. Stability can also be investigated for linear FO systems in their state space form.

Root locus can be used to examine the stability of a given closed-loop (CL) system when k is increased from 0 to ∞. Note that for $k=0$, the open loop and closed loop poles coincide. The RL method of Evans [15], though heuristic in nature, is one of the most popular and powerful geometric methods for both analysis and design of single-input single-output (SISO) IO linear time-invariant (LTI) systems. Consider a feedback system as shown in Fig. 1, here an amplifier with gain k (which can be varied from 0 to ∞) is connected in cascade with the plant $G(s)$. From [15] we have, $kG(s)H(s)$ as an Open Loop Transfer Function (OLTF), $kG(s)/(1+ kG(s)H(s))$ is the Closed Loop Transfer Function (CLTF), and $1+kG(s)H(s)$ is the characteristics equation.

Further advances in RL are presented in [16–18]. RL method of IO system analysis can be used for analysis of linear FO systems.

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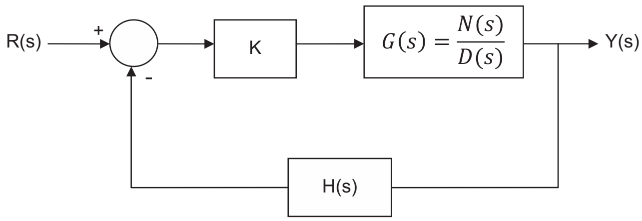

Fig. 1. Block diagram representation of closed-loop linear FO system.

Merrikh Bayat et al. in [13] extended the RL for FO systems. Important features of the classical RL method such as asymptotes, roots condition on the real axis, and breakaway points were extended to fractional case. The method is good but involves substantial calculations. Machado in [19] developed an algorithm for construction of RL. He adopted the method of evaluation of roots within two stage grid. It gives good performance for the RL of different types of systems and primarily plots RL on principal Riemann sheet. However the algorithm seems to be complicated and further no analysis is done from the plots. In [20] Abir-De et al. showed analytical method to draw RL of commensurate FO systems. They plotted the RL in the transformed w-plane and for analysis purpose it is again transformed back to the s-plane and only the real axis locus is considered for analysis. A good effort for studying the classical features of RL such as asymptotes, angle of departure is made in [20]. In all these papers the fact that only the roots on Principal Riemann Sheet (PRS) are responsible for different dynamics [21] is used and RL on only principal Riemann sheet is plotted and analyzed. The issue of RL branches in secondary Riemann sheets is not addressed. But as given later in this paper, one needs to keep a track of the RL branches originating from the poles in secondary sheets. This is because for some values of gain k, these branches may enter the stable/unstable regions of PRS. This will help to know a more precise range of gain k for which the system remains stable or becomes unstable. In this work a new approach is proposed to plot RL which is quite simple making the stability analysis of FO systems very easy and straightforward. The results obtained by plotting RL using the proposed method are verified using step response.

In this paper we plot RL in the transformed w-plane. RL is plotted by simply using the `rlocus` subroutine for IO systems in MATLAB [22]. Due to this we can directly use the features of plotting RL in MATLAB. We get directly the values of $k_{marginal}$ from the plot. The reliability of RL is verified by analytically finding the marginal values of k when RL cuts the imaginary axis ($k_{marginal}$). RL is plotted in transformed plane which includes all the Riemann sheets and the RL branches. With this plot that includes RL branches on the secondary Riemann sheets and the stability criteria of FO systems discussed in section 3, we can easily find whether these branches enter the stable or unstable region of principal Riemann sheet at some value of gain k. This is used to determine the range of $k_{marginal}$. For example, it may happen that one RL branch is entering the stable region for a value of gain k and simultaneously other RL branch from secondary Riemann sheet may be entering the unstable region, resulting in the instability of the system, may be for all values of gain k.

The main contributions of paper are as follows: (1) Proposing a novel method for plotting the RL of linear FO systems. This method is very easy to use and involves transformation of the FO system to its IO counterpart. (2) Plotting the RLs for a variety of FO systems with different structures and pole-zero locations. (3) Giving a detailed analysis of the CL stability of the FO system using the proposed RL technique.

The paper is organized as follows: Section 2 discusses the mathematical preliminaries for fractional calculus, Section 3 discusses the stability of FO systems using Riemann surfaces, Section 4 describes the proposed method of RL, various FO systems with their RL plots are analyzed in Section 5, and Section 6 gives the conclusion for the work done.

2. Fractional calculus

This section discusses fractional derivatives and transfer function model of FO systems.

2.1. Definitions of fractional derivatives

The three equivalent definitions most frequently used for the general fractional derivatives (FD) are the Grunwald–Letnikov (GL) definition, the Riemann–Liouville and the Caputo definition [23]. In all the definitions below, the function $f(t)$ is assumed to be sufficiently smooth and locally integrable.

1. *The Grunwald–Letnikov definition* of FO, $\alpha \in \mathcal{R}$, using Podlubny's limited memory principle [24] is given by

$$ {}_aD_t^{\alpha} f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{[t-a/h]} (-1)^j \, {}^{\alpha}C_j f(t-jh), \tag{1}$$

   where $[\cdot]$ means the integer part and ${}^{\alpha}C_j$ is the binomial coefficient.
2. *The Riemann–Liouville definition* obtained using the Riemann–Liouville integral is given by

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{2}$$

   for $(n-1<\alpha<n)$ and $\Gamma(\cdot)$ is the Gamma function.
3. *The Caputo definition* can be written as

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{3}$$

   for $(n-1<r<n)$, where $f^n(\tau)$ is the nth-order derivative of the function $f(t)$.

Since we deal with causal systems in the control theory, the lower limit is fixed at $a=0$ and for the brevity it will not be shown in this paper. We see that the Caputo definition is more restrictive than the Riemann–Liouville. Nevertheless, it is preferred by engineers and physicists because the FDEs with Caputo derivatives have the same initial conditions as that for the integer-order differential equations. Note that the FDs calculated using these three definitions coincide for an initially relaxed function.

2.2. Linear FO systems

A general FO system can be described by an Fractional Differential Equation (FDE) of the form:

$$ a_nD^{\alpha_n}y(t) + a_{n-1}D^{\alpha_{n-1}}y(t) + \cdots + a_0D^{\alpha_0}y(t) = b_mD^{\beta_m}u(t) + b_{m-1}D^{\beta_{m-1}}u(t) + \cdots + b_0D^{\beta_0}u(t), \tag{4}$$

where a_k, $(k=0,\ldots,n)$, b_k, $(k=0,\ldots,m)$ are constant, and α_k, $(k=0,\ldots,n)$, β_k, $(k=0,\ldots,m)$ are arbitrary real or rational numbers and without loss of generality they can be arranged as $\alpha_n > \alpha_{n-1} > \cdots > \alpha_0$, and $\beta_m > \beta_{m-1} > \cdots > \beta_0$. Here $D^{\gamma} \equiv {}_0D_t^{\gamma}$ denotes the Riemann–Liouville or Caputo fractional derivative [21], the FDE given in Eq. (4) is expressed in the form:

$$ G(s) = \frac{b_ms^{\beta_m} + \cdots + b_1s^{\beta_1} + b_0s^{\beta_0}}{a_ns^{\alpha_n} + \cdots + a_1s^{\alpha_1} + a_0s^{\alpha_0}} = \frac{Q(s^{\beta_k})}{P(s^{\alpha_k})}, \tag{5}$$