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Analysis of an explicit and matrix free fractional step method for incompressible flows

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Dedicated to Professor John Argyris, a respected scientist

Abstract

In this paper we present the analysis of an explicit and matrix free fractional step method for incompressible flows. The presented method can be employed in either conservation or non-conservation form. The stabilization, convergence and conservation aspects of the presented method are discussed. A procedure for eliminating the first order error in time introduced by the split is proposed. Some benchmark steady and unsteady state examples are presented to demonstrate the proposed new aspects of the matrix free scheme.

Keywords: Fractional step method; Characteristic based; Incompressible flows; Matrix free formulation; Dual time stepping; Artificial compressibility; Finite element method

1. Introduction

Over the last few years interest in developing accurate schemes for solving large scale incompressible flow problems has increased due to the emerging interdisciplinary applications such as fluid–structure interaction in biomedical studies. Several finite elements based incompressible fluid dynamics algorithms have been developed under the umbrella of stabilized methods. These methods include streamline upwind Petrov–Galerkin (SUPG), Galerkin least squares (GLS), finite calculus (FIC) and more recently subgrid scale (SGS) approach [1–6].

Another family of stabilized schemes, which are based upon higher order time stepping approach have been very popular in aerospace applications. The Taylor Galerkin (TG) and characteristic Galerkin (CG) methods are the two major methods under this category [1,7–15].

In all the above methods pressure stability becomes essential, when these schemes were employed to solve incompressible flow problems. One of the very popular procedures of dealing with the pressure instability in the incompressible flow context is *fractional step* or *projection method*. Recent works of the authors have been on combining the convection stabilization developed via higher order time stepping with a fractional step method. The resulted method is widely known as the characteristic based split (CBS) scheme [13–15].

The characteristic based methods have been developed over the last thirty five years and have now been widely employed to solve both convection–diffusion and Navier–Stokes equations [16–22]. The fractional step methods are not new and have been the subject of discussion since its introduction by Chorin [23]. However, the difference in this paper is that the method

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employed here is matrix free and fully explicit for steady state problems [24–28]. The popular advantages of an explicit scheme, such as easy to paralyze codes and smaller memory requirement, are well known.

The basis of the characteristic based spit (CBS) scheme and its applications have been discussed in many of the articles published in the past [13–15,24–36]. Comparison of the CBS scheme with other schemes is discussed in Refs. [37,38]. Our objective here is, thus, not to review the CBS scheme in detail but to analyse the matrix free CBS scheme for incompressible flows. We limit our discussion to laminar flows and also to linear triangular elements.

The explicit fractional step scheme based on an artificial compressibility was introduced in Ref. [24]. The method was developed by combining a classical fractional step method with the artificial compressibility scheme. The characteristic based stabilization was adopted to reduce oscillations in convection dominated flows. The pressure stabilization was achieved by introducing fractional stages into the solution process. In the approximation, the incompressible flow equations were always used in their conservation form and the deviatoric stresses were used in full. However, such an approximation is more expensive than using simplified incompressible flow equations. Also, the classical fractional step method used in the development of the original CBS scheme is known to introduce a first order time error in pressure solution if the pressure is completely removed at the fractional stage from the momentum equations [39–42]. In this paper, we consider both these issues. However, we give more emphasis to the elimination of the first order time error, when matrix free solution procedure is employed.

The fractional step methods can be derived from a semi- or fully-discrete form of the Navier–Stokes equations. We do provide the derivation of the explicit fractional step method using both approaches. However, we mainly use the classical form of the method derived from semi-discrete equations. The error introduced by the difference in semi- and fully-discrete form of approximations will be the basis for stabilizing the pressure of the second order fractional step method. A similar approach is proposed in Ref. [43] for quasi-implicit fractional step methods. For transient flow problems, implementation of such a stabilization will be achieved via a dual time stepping approach.

The paper is organized into following sections. In Section 2 we summarise the incompressible Navier–Stokes equations in various forms and their non-dimensional scales. Section 3 describes the matrix free fractional step method in some detail. All the relevant aspects of the scheme, including eliminating first order time error and dual time stepping are described in this section. Increasing the pressure stability of the second order scheme is presented in Section 4. Some numerical examples are presented in Sections 5 and 6. Finally, Section 7 draw some conclusions on the presented analysis.

2. Problem statement

The Navier–Stokes equations in conservation form may be written as

Mass conservation

$$\frac{\partial \rho}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t} = -\frac{\partial U_i}{\partial x_i}, \tag{1}$$

where c is the speed of sound and approaches infinity for incompressible flows. Thus, we replace this with a finite artificial compressible wave speed, β, as

$$\frac{1}{\beta^2}\frac{\partial p}{\partial t} = -\frac{\partial U_i}{\partial x_i} \tag{2}$$

to retain the time term in the equation. In the above equation,

$$U_i = \rho u_i, \tag{3}$$

with ρ being the density and u_i being the velocity components.

Momentum conservation

$$\frac{\partial U_i}{\partial t} = -\frac{\partial}{\partial x_j}(u_j U_i) + \frac{\partial \tau_{ij}}{\partial x_j} - \frac{\partial p}{\partial x_i}, \tag{4}$$

where τ_{ij} are the deviatoric stress components given by

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right), \tag{5}$$

where μ is the dynamic viscosity. The problem statement is complete with the following boundary conditions

$$\phi_i = \bar{\phi}_i \text{ on } \Gamma_\phi \quad \text{and} \quad q = \bar{q} \text{ on } \Gamma_f \tag{6}$$

in which

$$\Gamma = \Gamma_\phi \cup \Gamma_f, \tag{7}$$