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# Identifiability of fractional order systems using input output frequency contents

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ABSTRACT

In this paper, issues related to the identifiability of a fractional order system having its input and output frequency contents are discussed. The effects of the commensurate order $\alpha$ in the identifiability of the model structure and model parameters are analytically studied. It is shown that both identifiabilities (model structure and model parameters) are reduced remarkably for smaller values of $\alpha$. This phenomenon is observed even though the input signals are rich enough and system belongs to the model set. Our understanding is that the problem arises since differences among different members of the model set fall beyond the practically recognizable precision range. The issue is more problematic when $\alpha$ is smaller and measurements are noisy.



## 1. Introduction

Fractional calculus is a mathematical topic with a more than 300 year old history. There are multiple viewpoints of the fractional order system applications studied in the literature, such as stability analysis [1,2], system identification [3,4], system approximation [5], control [6,7], synchronization [8,9], dynamical behavior analysis [10–13], and so on.

System identification is an important part of control engineering and can be performed either in the time or frequency domain [14]. The first work in a deterministic approach of the frequency domain identification was reported in [15], where the existing nonlinear least squares problem is replaced by a linear least squares one by multiplying the equation error with the denominator of the transfer function. The authors in [16] overcame the lack of sensitivity to low frequency errors of the linear least squares estimator by an iterative procedure. In [17] and [18], the authors have solved the existing nonlinear least squares problem using the Newton–Gauss iteration scheme for a continuous and discrete time model, respectively. The identification of fractional models in a way rather close to [15] i.e. restricted to an all poles commensurate transfer functions, has been performed in [4]. A similar method for an all poles complex commensurate order system has been reported in [19]. The proposed technique was improved and generalized in [20] to include transfer functions with both zero(s) and pole(s). Some of the frequency domain identification methods for integer order models were extended to fractional ones in [21].

Identifiability can be considered both for structure and parameters of model set components. Studies on the structural identifiability of a model set can be traced for example in [22–29] and parameter identifiability for model set components has been discussed in references such as [27,30–32]. All these references deal with integer order systems. The problem of identifiability in fractional order models has not been noticed so far and to our best knowledge this paper is the first reported work on the general aspect of this subject.

From results reported in [20] it is revealed that the identifiability is lost for smaller commensurate order $\alpha$. In other words, different members of a model set, including the actual system, generate almost the same output frequency content for an input signal having extensive frequency content even in a wide frequency range, while they have completely different order combination and also different parameters. The number of different models with the same characteristics increases rapidly with a decrease in the value of $\alpha$.

The remainder of the paper is organized as follows. In Section 2 a brief discussion on the frequency domain identification based on order distribution is provided. Effects of the commensurate order $\alpha$ in the identifiability are discussed in Section 3. Illustrative examples in this regard are given in Section 4 to shed light on the problem appearing to worsen with a decrease in $\alpha$. In Section 5 some remarks on how to choose an acceptable model from the large number of estimated ones are presented. Some comments on the identifiability of the model structure and model parameters are given in Section 6. Finally the paper is concluded in Section 7.

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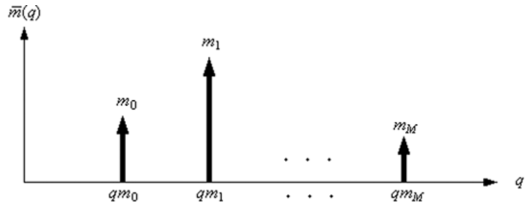


**Fig. 1.** Order distribution of the numerator $\bar{m}(q)$.

## 2. Frequency domain identification using order distribution

Allowing a restriction on the maximum possible numerator and denominator orders, a general system representation becomes

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\int_{qm_0}^{qm_M} \bar{m}(q)s^q \mathrm{d}q}{\int_0^{qn_N} \bar{n}(q)s^q \mathrm{d}q} = \frac{m_M s^{qm_M} + m_{M-1}s^{qm_{M-1}} + \cdots + m_1 s^{qm_1} + m_0 s^{qm_0}}{n_N s^{qn_N} + n_{N-1}s^{qn_{N-1}} + \cdots + n_1 s^{qn_1} + 1} \quad (1)$$

where $qm_0$ is the lower limit of the numerator and $qm_M$, $qn_N$ are the upper limits on the differential orders. Notice that the constant coefficient of the denominator polynomial is 1.

$\bar{m}(q)$and $\bar{n}(q)$ are the order distribution of the numerator and denominator respectively (as shown in Fig. 1 for $\bar{m}(q)$). In the discrete case, the order distribution contains Dirac-delta functions at distinct orders of $\bar{m}(q)$ or $\bar{n}(q)$.

Now we choose $L$ frequency samples $\omega_i$ $(s = j\omega)$ which are linearly spaced in $[\omega_{\min}, \omega_{\max}]$. Assume that the measurements have been obtained from the following system representation

$$Y(j\omega) = G(j\omega)U(j\omega) + E(j\omega), \quad (2)$$

where diagonal matrices $U(j\omega)$, $Y(j\omega)$, and $E(j\omega)$ are input, output, and noise respectively and are defined as $U(j\omega) = \mathrm{diag}([U(j\omega_1), \ldots, U(j\omega_L)])$, $Y(j\omega) = \mathrm{diag}([Y(j\omega_1), \ldots, Y(j\omega_L)])$, $G(j\omega) = \mathrm{diag}([G(j\omega_1), \ldots, G(j\omega_L)])$, and $E(j\omega) = \mathrm{diag}([\varepsilon(j\omega_1), \ldots, \varepsilon(j\omega_L)])$. Let us define

$$W_m = \begin{bmatrix} (j\omega_1)^{qm_0} & (j\omega_1)^{qm_1} & \ldots & (j\omega_1)^{qm_M} \\ (j\omega_2)^{qm_0} & (j\omega_2)^{qm_1} & \ldots & (j\omega_2)^{qm_M} \\ \vdots & \vdots & \ddots & \vdots \\ (j\omega_L)^{qm_0} & (j\omega_L)^{qm_1} & \ldots & (j\omega_L)^{qm_M} \end{bmatrix},$$

$$W_n = \begin{bmatrix} 1 & (j\omega_1)^{qn_1} & \ldots & (j\omega_1)^{qn_N} \\ 1 & (j\omega_2)^{qn_1} & \ldots & (j\omega_2)^{qn_N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (j\omega_L)^{qn_1} & \ldots & (j\omega_L)^{qn_N} \end{bmatrix},$$

$$m = \begin{bmatrix} m_0 \\ \vdots \\ m_M \end{bmatrix}, \qquad n = \begin{bmatrix} n_1 \\ \vdots \\ n_N \end{bmatrix}, \qquad n' = \begin{bmatrix} 1 \\ n \end{bmatrix},$$

and

$$S = W_m^H |U|^2 W_m, \qquad T = W_m^H U^* Y W_n, \qquad R = W_n^H |Y|^2 W_n, \quad (3)$$

$$R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad T = \begin{bmatrix} f & g \end{bmatrix},$$

where $a, b, c, d, f$ and $g$ are, respectively, $1 \times 1$, $1 \times N$, $N \times 1$, $N \times N$, $(M+1) \times N$, and $(M+1) \times 1$ matrices.

To determine the unknown coefficients $n$ and $m$, the following weighted quadratic function of the estimation error in (2) is considered.

$$J \triangleq (YW_n n' - UW_m m)^H (YW_n n' - UW_m m) = n'^H W_n^H E^H E W_n n'. \quad (4)$$

This choice reduces nonlinear programming appearing when $\bar{J}(\triangleq \ E^H E)$ is selected to standard quadratic programming. By applying the least squares technique on minimizing $J$, one obtains

$$\begin{cases} n = \left\{Re(d) - Re(f^T)\,[Re(S)]^{-1}\,Re(f)\right\}^{-1} \\ \qquad \times \left\{Re(f^T)\,[Re(S)]^{-1}\,Re(g) - Re(c)\right\} \\ m = [Re(S)]^{-1}\,[Re(f)n + Re(g)]\,. \end{cases} \quad (5)$$

One may use pseudo inverse algorithms for matrix inversions in (5) to eliminate the possible ill conditioning.

Let define sets $qn$ and $qm$ as follows

$$qn = \{0, qn_1, \ldots, qn_N\}, \qquad qm = \{qm_0, qm_1, \ldots, qm_M\} \quad (6)$$

and consider sets $Q_n$ and $Q_m$ such that $qn \subseteq Q_n$ and $qm \subseteq Q_m$, i.e. the model set contains the system. To estimate the correct values of $m$ and $n$, one may generate all subsets of $Q_m$ and $Q_n$ and then using (5) for each combination, find values of $m$ and $n$ that minimize the cost function $J$.

## 3. Effect of the commensurate order $\alpha$ on the model estimation

At first we build up some mathematical background for our later use.

**Theorem 1.** *A fractional order system with the commensurate order $\alpha$ is assumed as $F(s) = G(s^\alpha)$. The frequency response $F(j\omega) = G((j\omega)^\alpha)$ could be written as $G(a + jb)$ where $a = \omega^\alpha \cos(0.5\alpha\pi)$ and $b = \omega^\alpha \sin(0.5\alpha\pi)$.*

**Proof.** Let us define $(j\omega)^\alpha = a + jb$. Then replace $(j\omega)^\alpha$ by $j^\alpha \omega^\alpha = e^{j0.5\alpha\pi}\omega^\alpha$ which is equivalent to $\omega^\alpha \cos(0.5\alpha\pi) + j\omega^\alpha \sin(0.5\alpha\pi)$. By equating the real and complex parts of both sides of the first equation the proof of the theorem is completed. □

**Corollary 1.A.** *The locus of the points as $s = a + jb$, which is related to a fractional system with the commensurate order $\alpha$, is the line passing through the origin with slope $\tan(0.5\alpha\pi)$.*

**Corollary 1.B.** *As a physical meaning, $G(a + jb)$ is the frequency response of a fractional system with the commensurate order $\alpha$ at frequency $\omega$, that is $F(j\omega) = G((j\omega)^\alpha)$ where $\alpha = \frac{2}{\pi}\tan^{-1}\left(\frac{b}{a}\right)$ and $\omega = (a^2 + b^2)^{\frac{1}{2\alpha}}$.*

**Corollary 1.C.** *For $a = 0$, the fractional order system would be an integer order one defined by $G(j\omega)$ where $\omega = b$.*

**Theorem 2.** *Assume $Q(s)$ has the following bounds over the line $s = (\cot(0.5\alpha\pi) + j)b$ for $b_1 \le b \le b_2$*

$$A_1 \le |Q(s)| \le A_2 \quad \text{and} \quad P_1 \le Arg[Q(s)] \le P_2. \quad (7)$$

*To have the frequency response of $P(s) = Q(s^\alpha)$ in the same range i.e.*

$$A_1 \le |P(j\omega)| \le A_2 \quad \text{and} \quad P_1 \le Arg[P(j\omega)] \le P_2, \quad (8)$$

*$\omega$ should be limited to*

$$\left(\frac{b_1}{\sin(0.5\alpha\pi)}\right)^{\frac{1}{\alpha}} \le \omega \le \left(\frac{b_2}{\sin(0.5\alpha\pi)}\right)^{\frac{1}{\alpha}}. \quad (9)$$

**Proof.** According to Theorem 1, $b = \omega^\alpha \sin(0.5\alpha\pi)$ holds. Therefore inequality (9) is an obvious consequence of $b_1 \le b \le b_2$. □