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Two-dimensional ARMA model order determination

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ABSTRACT

Determination of the order of a model is the key first step towards modeling any dynamic systems, particularly two-dimensional processes. In this paper, a new method for two-dimensional (2-D) Gaussian ARMA model order determination is proposed. In the proposed method, the AR and MA orders are first independently determined, then the procedure for model order determination of the 2-D ARMA model is outlined. The model is assumed to be causal, stable, linear, and spatial shift-invariant with $p_1 \times p_2$ quarter-plane (QP) support. Numerical simulations are presented to show the effectiveness of the proposed new approach.

1. Introduction

Over the last three decades, there has been considerable interest in modeling two-dimensional (2-D) signals by 2-D autoregressive moving-average (ARMA) models. These models are used in several areas such as 2-D modeling [1,2], spectral estimation [3,4], 2-D system identification and parameter estimation [5–8]. In most cases, the model order is assumed to be known. In other words, all works in the 2-D case have focused on the problem of parameter estimation [6–9].

In most realistic situations, however, the model order is not known *a priori* and must be determined before the parameter estimation.

During the last three decades, several new methods and algorithms have been proposed for one-dimensional (1-D) model order selection; however, 2-D model order selection has not received as much attention.

The 1-D model order determination problem has been studied for a long time, by many researchers such as Akaike [10], Rissanen [11], Chan and Wood [12], Giannakis and Mendel [13], Zhang et al. [14,15], Liang et al. [16], Xiao et al. [17], Al-Smadi et al. [18,19], Gelach et al. [20], Ridder et al. [21], Pappas et al. [22,23], etc.

The existing order determination methods can be divided into two categories, namely, information theoretic criterion methods and linear algebraic methods.

Information theoretic criterion methods, e.g., AIC [10], minimum describing length (MDL) [11], and minimum eigenvalue (MEV) criterion [16], are evaluated by minimizing an expression that depends on the number of parameters. The value of order that yields the minimum value of the selected criterion is chosen as the best estimate of the true model order. The linear algebraic methods are based on determinant and rank testing algorithms, SVD-based methods, etc. In these methods, the order of the system is usually determined using the rank of special matrices. [24–27] are examples of this class.

In this paper, a new method for model order determination of 2-D ARMA models is described. First, using a combination of MDL criterion and instrumental variable (IV) method, 2-D AR order selection of a 2-D ARMA model is determined. After selecting the AR order, the MA order is determined via the SVD of a correlation matrix. In this work, the AR and MA orders of a 2-D ARMA model are first independently determined, then the complete procedure is outlined.

The paper is organized as follows: The problem formulation and the basic algorithm are described in Sections 2 and 3, respectively. Section 4 provides numerical simulations in order to illustrate the effectiveness of the proposed method. Section 5 concludes the paper.

2. Problem formulation for 2-D ARMA model order estimation

Consider a 2-D causal, stable, linear, and spatial shift-invariant ARMA model defined by [24]

$$\sum_{i=0}^{p_1^*}\sum_{j=0}^{p_2^*} a_{i,j} y_{t_1-i,t_2-j} = \sum_{i=0}^{q_1^*}\sum_{j=0}^{q_2^*} b_{i,j} e_{t_1-i,t_2-j}. \quad (1)$$

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The coefficients $a_{i,j}$ and $b_{i,j}$ are the autoregressive (AR) and moving-average (MA) parameters, respectively. The order of the AR part is (p_1^*, p_2^*), while the order of the MA part is (q_1^*, q_2^*).

In this work, the effect of noise in the measured output data has not been considered. The following conditions are assumed to hold.

Assumption 1. $a_{0,0} = 1$, and the sequence e_{t_1,t_2} is a zero-mean white noise process of variance σ_e^2.

Assumption 2. $a_{p_1^*,p_2^*} \neq 0$, $b_{q_1^*,q_2^*} \neq 0$.

The region of support (ROS) is the neighbor set of the model whose shape determines the causality of the model [28].

Since the true orders (p_1^*, p_2^*) and (q_1^*, q_2^*) are unknowns, the general format of (1) with $(p_1^*, p_2^*; q_1^*, q_2^*)$ replaced by unknowns $(p_1, p_2; q_1, q_2)$ is considered.

3. 2-D ARMA model order determination

The proposed algorithm has three parts, first, AR order determination, second, MA order determination and third, 2-D ARMA order determination algorithm. In this section, these parts are described.

3.1. Algorithm for AR order estimation

Assuming that the data length is $N_1 \times N_2$ (that is $t_1 = 0, 1, 2, \ldots, N_1 - 1, t_2 = 0, 1, 2, \ldots, N_2 - 1$), the Eq. (1) can be rewritten in a matrix form as follows:

$$Y\theta = W. \tag{2}$$

In the above equation, Y is an $(N_1N_2) \times (p_1 + 1)(p_2 + 1)$ output data matrix, θ is a $(p_1 + 1)(p_2 + 1) \times 1$ parameter vector, and W is an $(N_1N_2) \times 1$ input data vector.

$$\theta = [a_{0,0} \ldots a_{0,p_2} a_{1,0} \ldots a_{1,p_2} \ldots a_{p_1,0} \ldots a_{p_1,p_2}]^{\mathrm{T}} \tag{3a}$$

$$W = [w_{0,0} w_{0,1} \ldots w_{0,N_2-1} w_{1,0} w_{1,1} \ldots w_{1,N_2-1} \ldots w_{N_1-1,0} \ldots w_{N_1-1,N_2-1}]^{\mathrm{T}} \tag{3b}$$

where

$$w_{t_1,t_2} = \sum_{i_1=0}^{q_1} \sum_{i_2=0}^{q_2} b_{i_1,i_2} e_{t_1-i_1,t_2-i_2} \tag{3c}$$

and

$$Y = \begin{bmatrix} Y_0 & O & \cdots & O \\ Y_1 & Y_0 & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ Y_{N_1-1} & Y_{N_1-2} & \cdots & Y_{N_1-1-p_1} \end{bmatrix} \tag{3d}$$

$$Y_i = \begin{bmatrix} y_{i,0} & 0 & \cdots & 0 \\ y_{i,1} & y_{i,0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ y_{i,N_2-1} & y_{i,N_2-2} & \cdots & y_{i,N_2-1-p_2} \end{bmatrix}$$

$$i = 0, 1, \ldots, N_1 - 1 - p_1. \tag{3e}$$

Note that O in (3d) is a zero matrix with dimension $N_2 \times (p_2 + 1)$. An instrumental variable (IV) matrix can be defined as

$$Z = \begin{bmatrix} Z_0 & O & \cdots & O \\ Z_1 & Z_0 & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ Z_{N_1-1} & Z_{N_1-2} & \cdots & Z_{N_1-1-k_1} \end{bmatrix} \tag{4a}$$

where

$$Z_i = \begin{bmatrix} z_{i,0} & 0 & \cdots & 0 \\ z_{i,1} & z_{i,0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ z_{i,N_2-1} & z_{i,N_2-2} & \cdots & z_{i,N_2-1-k_2} \end{bmatrix} \tag{4b}$$

$$i = 0, 1, \ldots, N_1 - 1 - k_1$$
$$k_1 > p_1, k_2 > p_2$$

where z_{t_1,t_2} is an IV sequence. Several choices of z_{t_1,t_2} are possible as long as IV sequence is uncorrelated with the noise part w_{t_1,t_2} and fully correlated with the observed part y_{t_1,t_2} [29].

In this paper, z_{t_1,t_2} emerged from the delayed observed data $y_{t_1-l_1,t_2-l_2}$ with $l_1 > q_1, l_2 > q_2$. Premultiplying (2) by $\frac{1}{N_1N_2}Z^{\mathrm{T}}$ and considering $V = \frac{1}{N_1N_2}Z^{\mathrm{T}}W$, the following equation is obtained

$$\frac{1}{N_1N_2}Z^{\mathrm{T}}Y\theta = V \tag{5}$$

V is an asymptotically Gaussian distribution with zero mean [17]. If D is defined as $D = \frac{1}{N_1N_2}Z^{\mathrm{T}}Y$, then Eq. (5) can be rewritten as

$$D\theta = V. \tag{6}$$

In which, dimensions of D, θ, and V are respectively $(k_1 + 1)(k_2 + 1) \times (p_1 + 1)(p_2 + 1)$, $(p_1 + 1)(p_2 + 1) \times 1$, and $(k_1 + 1)(k_2 + 1) \times 1$. Now, the $(p_1 + 1)(p_2 + 1) \times (p_1 + 1)(p_2 + 1)$ matrix R is defined as

$$R = D^{\mathrm{T}}D. \tag{7}$$

Note that R is a symmetric and positive semi-definite matrix. The new method proposed in this paper permits the choice of the AR order of 2-D ARMA model in (1) with high accuracy without any parameter estimation. This method uses both 2-D MDL criterion and the matrix R.

In the 2-D case, the MDL order determination criterion appears as follows [28]:

$$J_{\mathrm{MDL}}(p_1, p_2) = -\log(f(V)) + \frac{1}{2}(p_1 + 1)(p_2 + 1) \times \log((k_1 + 1)(k_2 + 1)) \tag{8}$$

where $f(V)$ is the probability density function of V such that $V = [v_{0,0} \ldots v_{0,k_2} \ldots v_{k_1,0} \ldots v_{k_1,k_2}]^{\mathrm{T}}$. Since v_{t_1,t_2} is zero-mean white Gaussian noise,

$$f(V) = \frac{1}{(2\pi\sigma^2)^{\frac{(k_1+1)(k_2+1)}{2}}} \exp\left(-\frac{1}{2\sigma^2}V^{\mathrm{T}}V\right) = \frac{1}{(2\pi\sigma^2)^{\frac{(k_1+1)(k_2+1)}{2}}} \exp\left(-\frac{1}{2\sigma^2}\theta^{\mathrm{T}}R\theta\right) \tag{9}$$

where σ^2 is the variance of v_{t_1,t_2}. Replacing $f(V)$ by (9) into (8) results in:

$$J_{\mathrm{MDL}}(p_1, p_2, \theta) = \frac{(k_1 + 1)(k_2 + 1)}{2} \log(2\pi\sigma^2) + \frac{1}{2\sigma^2}\theta^{\mathrm{T}}R\theta + \frac{1}{2}(p_1 + 1)(p_2 + 1) \log((k_1 + 1)(k_2 + 1)). \tag{10}$$

Now, consider the following Lemma.

Lemma (*[28]*). *For fixed-order (p_1, p_2) and constraining θ to have unit Euclidean norm, the choice of θ that minimizes (10) is found to be the eigenvector associated with the minimum eigenvalue ($\lambda_{\min}$) of R.*