Journal of Electrostatics 84 (2016) 135–142

Contents lists available at ScienceDirect

Journal of Electrostatics

journal homepage: www.elsevier.com/locate/elstat

Some comments on the electrostatic forces between circular electrodes

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ARTICLE INFO

Article history:
Received 25 August 2016
Received in revised form
2 November 2016
Accepted 2 November 2016

Keywords:
Kelvin force
Coefficients of capacitance
Coefficients of potential
Asymptotic formulas for capacitance

ABSTRACT

We study the force between two circular electrodes in different configurations. A formula analogous to Kelvin's formula for the spheres is given in the case of equal disks held at the same potential and when one plate is earthed. An expression for the force at short distance between two arbitrarily charged disks is found: the generic case shows a logarithmic repulsive force, also for disks carrying charges of opposite sign. Some numerical computations support the results. A classification for the possible behaviors of the force is proposed on the basis of a decomposition of the capacitance matrix. It is shown that the forces depend strongly on the dimensionality of the contact zone between the conductors. The analysis is supported by a numerical computation carried for the case of two disks of different radii.

1. Introduction

The knowledge of the electrostatic force between two charged conductors has a theoretical and practical importance. Its exact calculation is possible through the capacitance coefficients [1], whose analytical value is known only for few selected geometries and arrangements of the electrodes.

For large distances between the charged bodies the problem can be understood in terms of the properties of the isolated conductors [2] and, in principle, it is solved. For short distances, the situation is more complex. A physical argument rests on the fact that when the two conductors touch, a new system is formed. Some unexpected properties arise in this regime. For example, two spheres of like charges almost always attract, as shown by J.Lekner [3].

A particularly intriguing problem is the following: which is the force between two equal conductors at the same potential? This problem was solved in a classical work by Lord Kelvin [4] for the case of two spheres of equal radius a finding a repulsive finite force at short distances:

$$F = \frac{Q^2}{a^2}\frac{4\log(2)-1}{24\,(\log(2))^2} \tag{1}$$

Another classical case studied by Kelvin is a couple of spheres when one of the two is earthed. These problems have been recently revisited and generalized to spheres of different radii by J.Lekner [3], who give an instructive study of different physical situations.

In this work we study the same problem for the case of two coaxial disks of radius a, when the distance ℓ between them goes to zero. For two disks with fixed charges Q_1, Q_2 we find a *logarithmically divergent* force at short distances, the only exception being $Q_2 = -Q_1$, when the force is attractive and constant. Moreover a curious, and potentially interesting, behaviour is observed in this near regime when the disks carry charges of different sign and of different absolute value. We found that the interaction cancels at a given distance and become repulsive for smaller gaps. In a way, the behaviour is the counterpart of that reported for two spheres. Owing to the redistribution of charges on the surface of the disks the interaction pass from attraction to repulsion for total charges of different sign. Surprisingly, this situation appears also for two disks with charges of the same sign if the ratio of the two charge is small, $Q_1/Q_2 \lesssim 0.016$.

With conductors we can consider configurations different from the standard case of two bodies of given charges, i.e. two conductors at fixed potentials and one conductor with charge Q while the other is held at fixed potential. These two cases cover the Kelvin's configurations. We will give below the small distance behaviour of the forces in these cases, for two disks.

This paper is organized as follows. In section 2 we present the

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http://dx.doi.org/10.1016/j.elstat.2016.11.001

general framework and give a formal expression for the forces. In section 3, a simple analysis of an integral equation provide us the needed asymptotic terms in the capacitance matrix. We complete the analysis and give the explicit expression for forces between the disks in different cases. Particularly, we show that disks charged with different charges of opposite sign repel each other for an appropriate reduction of the inter-electrode distance. The calculated distribution of surface charge in this case gives a physical insight in this curious result. In section 4 we propose a generalization to the case of arbitrary conductors, based on a particular decomposition of the capacitance matrix. With this method the role of the dimensionality of the contact zone is emphasized and the corresponding different behaviour of forces is naturally explained. This approach is checked in the case of two disks with different radii, where a constant force arises at short distances.

2. Method

Charges and potentials for two electrodes are linearly related

$$Q_i = \sum_j C_{ij} V_j; \quad V_i = \sum_j M_{ij} Q_j \tag{2}$$

C_{ij} are the elements the symmetric capacitance matrix and its inverse, with elements M_{ij}, is the potential matrix. The electrostatic energy of the system is given by the quadratic forms

$$W = \frac{1}{2}\sum_{i,j} C_{ij} V_i V_j = \frac{1}{2}\sum_{ij} M_{ij} Q_i Q_j. \tag{3}$$

In the general case, all the quantities in both quadratic forms of (3) depend on the distance ℓ between the conductors. On the other hand, the choice of the first or the other form is convenient when conductors at fixed potentials or at fixed charges are studied. The force between the conductors is found by differentiation of energy. For example, if the two disks have fixed charges, Q_1 and Q_2 respectively the force in the axial direction is

$$F = -\frac{1}{2}\sum_{i,j} Q_i Q_j \frac{\partial}{\partial \ell} M_{ij} \tag{4}$$

All the previous formulas simplify for equal conductors, e.g. equal disks, because $C_{11} = C_{22}$ and $M_{11} = M_{22}$. For equal conductors, it is convenient to distinguish in the capacitance matrix the usual relative capacitance, given in this case by $C = (C_{11} - C_{12})/2$ and the symmetric combination $C_g = (C_{11} + C_{12})$ which enters in the computation of the energy for fixed potentials. In terms of these quantity

$$C_{11} = C + \frac{1}{2}C_g; \quad C_{12} = -C + \frac{1}{2}C_g \tag{5}$$

Let us consider the electrodes at fixed charges. The matrix M_{ij} is calculated inverting C_{ij} and substituting in (3) we obtain for the energy

$$W = \frac{1}{4}(Q_1 + Q_2)^2 \frac{1}{C_g} + \frac{1}{8}(Q_1 - Q_2)^2 \frac{1}{C} \tag{6}$$

The force is given by

$$F(Q_1, Q_2, \kappa) = -\frac{(Q_1 + Q_2)^2}{4}\frac{\partial}{\partial \ell}\frac{1}{C_g} - \frac{(Q_1 - Q_2)^2}{8}\frac{\partial}{\partial \ell}\frac{1}{C} \tag{7}$$

where $\kappa = \ell/a$ is called aspect ratio of the two-disks system.

The rationale behind the decomposition (5) lies in the isolation of the divergent behaviour as $\ell \to 0$. The quantity $C_{11} + C_{22} + 2C_{12}$ is the total charge when the two bodies are held at unity potential and in the general case [5], it satisfies

$$\lim_{\ell \to 0}(C_{11} + C_{22} + 2C_{12}) = C_T \tag{8}$$

where C_T is the self-capacitance of the conductor obtained when the two separated bodies touch. For equal conductors $C_g \to C_T/2$ which is a finite quantity. Let us note that C_g is the only term which enter in the computation of the force for equal charges.

It is well known that the expression (7) is valid in every circumstance, for equal conductors, depending only on Coulomb's law, but the dependence on ℓ is hidden also in the charges when the bodies are held at fixed potential. For completeness let us shortly review the case of equal potential $V_1 = V_2 = V$. The charges are equal by symmetry $Q_1 = Q_2 = Q_V$ and from (7)

$$F_V = F(Q_V, Q_V, \kappa) = -Q_V^2 \frac{\partial}{\partial \ell}\frac{1}{C_g} = \frac{Q_V^2}{C_g^2}\frac{\partial}{\partial \ell}C_g \tag{9}$$

By noticing that from (2) in this case $Q_V = C_g V$ we can also write

$$F_V = V^2 \frac{\partial}{\partial \ell} C_g \tag{10}$$

This result can be directly obtained by expressing the electrostatic energy W in terms of V and taking $F = +\partial_\ell W$. The plus sign is due to the energy supplied by the voltage source, as explained in textbooks [6]. Using the decomposition (5) the general case of two different potentials V_1, V_2 give rise to a force

$$F_{V_1,V_2} = \overline{V}^2 \frac{\partial}{\partial \ell} C_g + \frac{\Delta V^2}{2}\frac{\partial}{\partial \ell} C \tag{11}$$

with $\overline{V} = (V_1 + V_2)/2$ and $\Delta V = V_1 - V_2$.

A third interesting configuration is one conductor with charge Q_1, fixed, and the second one held at potential 0. To give the explicit dependence of the force on ℓ we can follow the procedure used above. As $V_2 = 0$ the conductor 1 is at potential $V_1 = Q_1/C_{11}$ while conductor 2 has a charge $Q_2 = C_{21}V_1 = Q_1 C_{12}/C_{11}$. Substitution in (7) gives, after expressing C_g and C in terms of C_{11} and C_{12}:

$$F_E = -\frac{1}{2} Q_1^2 \frac{\partial}{\partial \ell}\frac{1}{C_{11}} \tag{12}$$

The same result can be obtained more easily from (3), but this derivation has the merit of eliminating any doubt about signs. Expressions (10) and (12) apply to Kelvin's configurations.

In summary, the knowledge of the force between the two conductors is equivalent to that of the dependence of the coefficients of capacitance and of potential, forming matrix **C** and **M**. Unfortunately, their analytic forms is limited to few cases. On the contrary, some concluding remarks can be drawn from the asymptotic behaviour of these parameters both in the far limit and in the near one. Elsewhere [2] we treated the large distance behaviour of C_{ij} and M_{ij} and in this limit the forces can be easily obtained by general formulas involving the self capacitances and other intrinsic parameters of the conductors. Our aim is to get an explicit formula for the forces in the case of two coaxial disks of radius a separated by a distance ℓ in the region $\ell \ll a$. Following (7) this purpose requires the knowledge of the behaviour of the relative capacitance C and of C_g in the near limit ($\kappa \to 0$) and this is the content of the following section.