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Closed-form expressions of the electrostatic potential close to a grid placed between two plates

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ABSTRACT

The potential distribution between a grid and two plates is an electrostatic problem already solved for various applications such as MultiWire detectors used in Nuclear Physics, or electrostatic precipitators in Engineering. Since references and notations for this analytical solution are ancient and rather bewildering, the first part of this paper presents a revisit and a discussion of the formulations that assume a line charge on the wire. This is completed by establishing a ready-to use closed-form expression valid for the general configuration where the plates are not grounded. The second part is about the investigation of the line model accuracy close to the wires, using both analytical and numerical approaches. In the symmetric case where the grid is placed at equal distances between two grounded plates, it is shown that the error can be modelled using a quadrupole charge. For the asymmetric case, a larger discrepancy of the line model is brought to light, with an error featuring a dipole-like distribution. In order to cope with this small but not negligible error, a classical dipole model is implemented, leading to an accurate theoretical expression of the potential.

1. Introduction

The potential distribution created by a conductive grid is a classical electrostatic problem solved several decades ago [1]. In engineering, its solution is used to model systems containing a grid, *e.g.* electrostatic precipitators [2]. Another application is the Multi-Wire Proportional Counter (MWPC) invented in the '70s for Nuclear Physics [3,4]. Many analytical techniques can be employed to solve this electrostatic problem: eigenfunctions expansion [5], conformal mapping [6,7], Green's functions [8], or special functions [1,3,8].

First, this paper proposes to revisit the closed-form expressions originating from works using different notations, formulations, or even languages [9,10]. Moreover, while expressions given in [3] only apply if the grid is placed between two grounded plates, our results are readily usable for an arbitrary configuration.

Secondly, this study investigates the accuracy of the formulas of the potential around the wire: actually, expressions given in [3] are derived from a line model of the electrodes, thus considering wires as lines instead of cylinders. The validity of this assumption is discussed in this paper. As a reference for precise determination, Laplace's equation is solved numerically using Comsol Multiphysics®, a commercial Finite Element Model (FEM) package.

Much of the current research on this topic focuses on the numerical approach [11], particularly useful when space charge effects are considered [12]. However, accurate analytical expressions are always necessary to design a prototype containing conductive grids, justifying the investigation presented herein.

The paper is organized as follows: in section 3, several formulas of the symmetric case (*i.e.* grid at equal distance between grounded plates) are recalled, and their compatibility and differences are discussed. Then, the general (asymmetric) case is considered in section 4, introducing a modern technique presented in [8] to solve the electrostatic problem. In section 5, the limitations of the line model used for MWPC are presented, using both theoretical and numerical approaches.

2. Notations and mathematical tools

The grid is constituted of parallel cylindrical wires along the z-axis with radius r_0 and pitch $s = 2a$. It is placed between two conductive parallel plates at a distance b from each other. The wire charge per unit length, λ, depends on the potential V_g applied to the grid and the potentials V_1, V_2 of the two plates.

To solve this 2D electrostatic problem, it is convenient to use the

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complex-valued variable $z = x + jy$, where x and y are the Cartesian coordinates, as well as the complex electrostatic potential $\Phi(z)$ such as the electric potential is $V(x, y) = \mathrm{Re}[\Phi(z)]$. It is recalled [1,10] that the complex electric field can be expressed as

$$\overline{E} = E_x - jE_y = -\frac{d\Phi}{dz} \tag{1}$$

In the case of a line charge placed at z_0 in free space, the complex potential is known to be $\Phi(z) = \frac{-\lambda}{2\pi\varepsilon_0}\log(z - z_0)$; the potential is then expressed as $V(x, y) = \frac{-\lambda}{2\pi\varepsilon_0}\log|z - z_0|$. It is noticeable that the argument of the logarithm function has a simple zero at the line location, property that will be fulfilled for the models presented in sections 3 and 4.

Two special functions are used in this paper, and some identities from [13] are provided in the Appendix. The Jacobi elliptic function $\mathrm{sn}(u,m)$ is doubly periodic for the complex variable u, with period $4K$ and $2K'$ for the real and imaginary axis, respectively. The nome is the quantity $q = \exp(-\pi K'/K)$, that is related to the parameter m through a method presented in the Appendix. The first Jacobi theta function can be noted $\theta_1(u|\tau)$ with the complex half-period $\tau = jK'/K$. This function also depends on $q = \exp(j\pi\tau)$ and can be expanded as an easy-to-compute Fourier series, as indicates Eq. (A.1).

3. Solution for the symmetric case

Many expressions can be found in literature for the symmetric case in which the grid is placed at mid-distance between two grounded plates. Three models [3,7,14] are recalled and discussed hereafter. Throughout this section, the origin is chosen on the grid, as illustrated in Fig. 1.

To facilitate the link with the original papers, both geometric parameters a and $s = 2a$ will be used throughout this section.

3.1. Expression with Jacobi elliptic sine function

In this subsection, the remarkable compact expression of Cooperman is presented [14]. Using a different orientation and origin, Tomitani proposed a somewhat more cumbersome formulation also using the same elliptic sine function [6]; we have verified that both models give identical numerical results. Note that the complex electric potential can be conveniently expressed using the Jacobi elliptic sine function $\mathrm{sn}(u, k)$. For this, it is recalled that this function has zeros at $u = 2nK$ and poles at $u = \pm jK' + 2nK$, with K the complete elliptic integral that depends on k. Therefore the function $\log|\mathrm{sn}|$ has positive and negative singularities, as pictured in Fig. 2.

The similarity between Figs. 1 and 2 suggests a simple correspondence between the complex potential Φ and the function $\log(\mathrm{sn})$, *i.e.* between the real potential V and the function $\log|\mathrm{sn}|$. In fact, elliptic functions are fully characterized (up to a constant

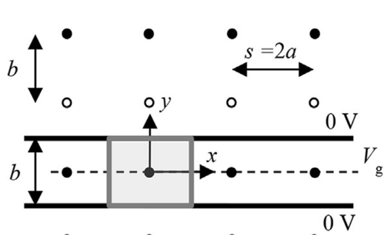

Fig. 1. Cross-section of the grid at mid-distance between two parallel plates. Positive and negative image sources are marked with ● and ○ respectively. The elementary domain is shaded.

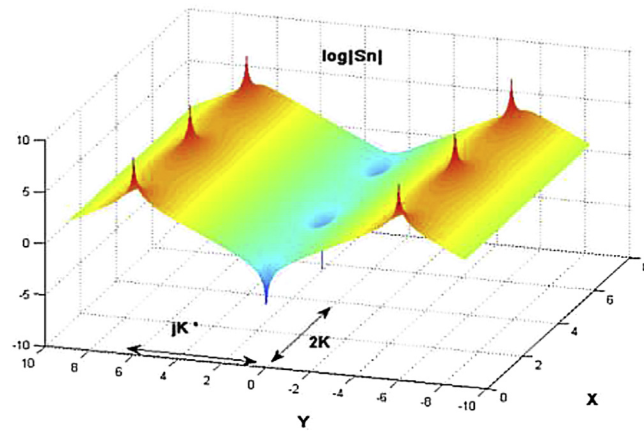

Fig. 2. Plot of $\log|\mathrm{sn}(u, k)|$ with $u = X + jY$ and $k = 0.01$.

multiplicative factor) by their poles and zeros, as well as their periods, property of complex analysis used in [1 vol 1 §4.3]. A linear variation is then assumed:

$$V = \frac{-\lambda}{2\pi\varepsilon_0}\log|\mathrm{sn}(\alpha z, k)| + V_3 \tag{2}$$

As the zeros of the sn function, *i.e.* the argument of the logarithm, must correspond to the wires locations, the scale coefficient α is found from

$$2\alpha a = 2K \tag{3}$$

The first image source being associated to $u = jK'$, it follows

$$\alpha b = K' \tag{4}$$

From the ratio b/a one can successively determine the nome,

$$q = \exp\left(-\pi\frac{K'}{K}\right) = \exp\left(-\pi\frac{b}{a}\right) \tag{5}$$

then, the elliptic integral K, and the modulus $m = k^2$. Boundary conditions on the grid and the plates are

$$\begin{cases} V\left(r_0 e^{j\theta}\right) = V_g \\ V(jb/2) = 0 \end{cases} \tag{6}$$

As $\alpha jb/2 = jK'/2$, the value indicated in Table A1 yields $\mathrm{sn}(\alpha jb/2, k) = jk^{-1/2}$; then, using the assumption $r_0 \ll a$ and Eq. (A.8), one obtains

$$V_{\mathrm{Coop}} = -\frac{C_{\mathrm{Coop}}V_g}{2\pi\varepsilon_0}\log\left|\sqrt{k}\,\mathrm{sn}(\alpha z, k)\right| \tag{7}$$

where $C_{\mathrm{Coop}} = \frac{\lambda}{V_g} = 2\pi\varepsilon_0/\log\frac{a}{Kr_0\sqrt{k}}$ is the capacitance per unit length.

Equation (7) is the exact solution of the electrostatic problem for the symmetric case with the assumption that wires are modelled as lines.

3.2. Symmetric case: expression with sine function

For negligible values of q, Erskine [3] gave the expression

$$V_{\mathrm{Ersk}} \simeq \frac{b\lambda}{4\varepsilon_0 s} - \frac{\lambda}{2\pi\varepsilon_0}\log\left|2\sin\frac{\pi z}{s}\right| \tag{8}$$