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Capacitance of an elliptical disk on a semi-infinite dielectric material

Mimi X. Yang [a], Fuqian Yang [*, b]

[a] Department of Electrical Engineering, Stanford University, Stanford, CA 94305-9505, United States
[b] Materials Program, Department of Chemical and Materials Engineering, University of Kentucky, Lexington, KY 40506, United States

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ABSTRACT

Using an integral transform, the mixed boundary value problem of a conducting, elliptical disk on a dielectric half-space in an electric field is reduced to the solution of an integral equation. An analytical expression of the electric system capacitance is derived, which is a function of the eccentricity of the elliptical disk. The electric charge and electric stress distribute non-uniformly over the surface of the elliptical disk and display local singularities at the edge of the elliptical disk. The square root singularity of the electric field at the edge of the elliptical disk leads to the divergent of the resultant force on the elliptical disk, which is physically unrealistic. There likely exist geometrical constraint and/or field constraint to limit the presence of the square root singularity of the electric field. For any symmetric conductor in an infinite space that consists of air (vacuum) and a semi-infinite dielectric material with symmetric plane being in the interface between the air and the dielectric material, the electric potential in the space is independent of the dielectric constant of the dielectric material.

1. Introduction

Progress in micro- and nanofabrication techniques has made it possible to fabricate surface structures of small scales on various substrates for applications in flexible electronics and bio-devices. Microstrip patches of a variety of geometries have been constructed as electronic components. There are extensive studies on the calculation of the capacitance of such patches with the focus on microstrip lines [1–3], circular patches [4–6] and elliptical patches [7–9]. Kuester [10] provided explicit approximations for the capacitance calculation of a microstrip patch of arbitrary shape.

Difficulty in analytically obtaining the electrostatic potential of a conducting elliptical patch on the surface of a dielectric substrate has led to the use of numerical methods to calculate the capacitance of elliptical patches. Boix and Horno [9] pointed out that there is little work on the capacitance analysis of elliptical patches and the results given by Sharma and Bhat [11] are dubious. They used variational techniques in the spectral domain to develop an algorithm for the calculation of a lower bound of the capacitance of a conducting, elliptical disk embedded in a lossless multilayered substrate. However, they did not provide an analytical result of the capacitance for the scenario of the ratio of the layer thickness to the major axis of the elliptical disk approaching infinity. Boix and Horno [8] also numerically calculated the modal capacitances and the gap capacitance of coupled microstrip elliptical disks embedded in layered media. Alad et al. [12] used the method of moments with triangular sub-areas to calculate the capacitance of an isolated elliptical plate and two parallel elliptical plates. It is worth mentioning that the capacitance of an isolated elliptical plate in an infinite medium of air (vacuum) was given by Lebedev et al. [13] and Liang et al. [14] using the ellipsoidal coordinate system and the Huygens principle, respectively.

It is known that microstrip antennas of circular and rectangular structures with multiple feeds can provide circular polarization [15]. However, a slightly elliptical radiator with a simple feed can also generate circular polarization. It is of practical importance to derive a closed-form solution of the capacitance of an isolated, elliptical disk on the surface of a dielectric half-space for better design of microstrip patches of elliptical shape for MMIC (monolithic microwave integrated-circuit) applications. In this work, the potential problem of a conducting, elliptical disk on a dielectric half-space is analyzed. Integral transforms are used to convert the boundary value problem to a solution of an integral equation. Analytical expressions of the distribution of electric charge on the surface of the conducting, elliptical disk and the capacitance are derived. The singular behavior of electric stress at the edge of the elliptical disk is discussed.

* Corresponding author.
E-mail address: fyang2@uky.edu (F. Yang).

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2. Formulation of the problem

Consider a conducting, elliptical disk which is placed on the surface of a semi-infinite dielectric material, as shown in Fig. 1. The major axis and minor axis coincide with the x-axis and y-axis of the (x, y, z) coordinate system, respectively, and the origin of the coordinate system is located at the center, O, of the elliptical disk. The semi-major axis and semi-minor axis of the elliptical disk are a and b, respectively, and the thickness of the elliptical disk is negligible. An electric potential of V is applied to the conducting, elliptical disk, and the bottom surface of the dielectric material is grounded.

The electric potential, φ, in the (x, y, z) coordinate system, satisfies the Laplace equation as

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0 \tag{1}$$

Let $x = a\rho\cos\theta$ and $y = b\rho\cos\theta$. The boundary conditions are

$$\varepsilon_r\frac{\partial\varphi_1}{\partial z}=\frac{\partial\varphi_2}{\partial z} \quad \text{for } \rho>1,\ \text{at } z=0 \tag{2}$$

$$\varphi_1=\varphi_2 \quad \text{for } \rho>1,\ \text{at } z=0 \tag{3}$$

$$\varphi_1\to 0 \text{ and } \varphi_2\to 0 \quad \text{for } |z|\to\infty \tag{4}$$

For $\rho < 1$, at $z = 0$,

$$\varphi_1=\varphi_2=V. \tag{5}$$

Here, φ_1 is the electric potential in the dielectric half-space, φ_2 is the electric potential in air, and ε_r is the relative dielectric constant of the dielectric half-space.

The charge density of the elliptical disk is calculated as

$$\rho_c=\varepsilon_0\varepsilon_r\frac{\partial\varphi_1}{\partial z}-\varepsilon_0\frac{\partial\varphi_2}{\partial z}, \tag{6}$$

which gives the total electric charge stored in the elliptical disk as

$$Q=\iint_\Omega \rho_c dxdy \tag{7}$$

in which the domain of Ω represents the area occupied by the elliptical disk, i.e. $\Omega = (x, y, 0)$: $x^2/a^2+y^2/b^2 < 1$.

3. Solutions of electric potential

Using the Fourier transform, the solutions of the electric potentials can be expressed as

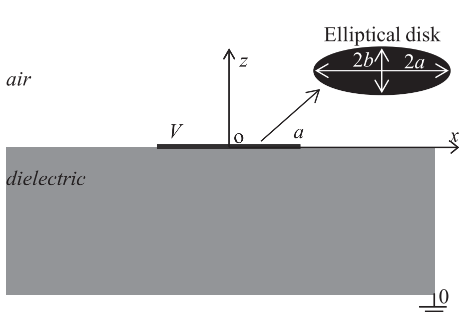

Fig. 1. Schematic of a conducting, elliptical disk on the surface of a dielectric half-space.

$$\varphi_1(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A_1(\xi,\eta)e^{i(\xi x+\eta y)+\lambda z}d\xi d\eta \quad \text{for } z<0 \tag{8}$$

$$\varphi_2(x,y,z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A_2(\xi,\eta)e^{i(\xi x+\eta y)-\lambda z}d\xi d\eta \quad \text{for } z>0 \tag{9}$$

with $\lambda=(\xi^2+\eta^2)^{1/2}$. Substituting Eqs. (8) and (9) in the boundary conditions of (3) and (5) yields

$$A_1(\xi,\eta)=A_2(\xi,\eta)\equiv A(\xi,\eta) \tag{10}$$

The boundary conditions of (2) and (5) then give

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{\xi^2+\eta^2}A(\xi,\eta)e^{i(\xi x+\eta y)}d\xi d\eta=0 \quad \text{for } \rho>1 \tag{11}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A(\xi,\eta)e^{i(\xi x+\eta y)}d\xi d\eta=V \quad \text{for } \rho<1 \tag{12}$$

To derive a closed-form solution of $A(\xi, \eta)$, let us define the function $F(x, y)$, which satisfies the following equation as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{\xi^2+\eta^2}A(\xi,\eta)e^{i(\xi x+\eta y)}d\xi d\eta=F(x,y) \quad \text{for } \rho<1 \tag{13}$$

From Eqs. (11) and (13), the inverse Fourier transform gives

$$A(\xi,\eta)=\frac{1}{2\pi}\frac{1}{\sqrt{\xi^2+\eta^2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(x,y)e^{-i(\xi x+\eta y)}dxdy \tag{14}$$

Substituting Eq. (14) in Eq. (12) yields

$$\iint_\Omega\frac{F(x',y')dx'dy'}{\sqrt{(x-x')^2+(y-y')^2}}=2\pi V \quad \text{for } \rho<1 \tag{15}$$

Note that the following equation is used in deriving Eq. (15).

$$\frac{1}{\sqrt{(x-x')^2+(y-y')^2}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-i\xi(x-x')-i\eta(y-y')}}{\sqrt{\xi^2+\eta^2}}d\xi d\eta \tag{16}$$

The solution of Eq. (15) is [13].

$$F(x,y)=\frac{V}{bK(k_0)}\frac{H(1-x^2/a^2-y^2/b^2)}{(1-x^2/a^2-y^2/b^2)^{1/2}} \tag{17}$$

with $H(\bullet)$ being the Heaviside unit function. The function $K(k_0)$ is the complete elliptic integral of the first kind as

$$K(k_0)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k_0^2\sin^2\theta}} \text{ and } k_0=\left(1-b^2/a^2\right)^{1/2} \tag{18}$$

Substituting Eq. (17) in Eq. (14), one obtains $A(\xi, \eta)$ as