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Reliable and efficient equilibrated a posteriori finite element error control in elastoplasticity and elastoviscoplasticity with hardening

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Abstract

This paper establishes the reliability and efficiency of equilibrated a posteriori estimates for L^2 stress error control of conforming displacement finite element approximations of incremental plasticity and viscoplasticity with hardening. Explicit expressions for upper bounds of the reliability constant that enters the guaranteed upper error bound illustrate its crucial dependence on the hardening material constants. Numerical experiments show that adaptive finite element solutions with marking strategy based on the max-refinement rule and local equilibrated error estimators lead to optimal empirical convergence rates.

Keywords: Elastoviscoplasticity; Hardening; Finite element method; A posteriori error estimate; Reliability; Efficiency; Adaptive algorithms

1. Introduction

Within each time step in a finite element analysis of elastoplastic and elastoviscoplastic evolution problems, one must solve a variational inequality with a complicated material law determined by admissible (generalised) stresses on top of the problem of linear elasticity. In the development of a posteriori error estimates for this class of problems one would expect that the estimates should involve also a measure of the residual in the material law, e.g., in some Kuhn–Tucker conditions on the plastic multiplier. However, as

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reported in [25], estimates of the norm of the residual in the equilibrium conditions might serve as an error estimator for reliable error control of finite element approximations of incremental plasticity and viscoplasticity. Corresponding residual-based estimates contain the term

$$\eta_{T,\mathrm{R}} = h_T^2 \int_T |f + \mathrm{div}_{\mathcal{T}}\, \sigma_h|^2 \mathrm{d}x + \int_{\partial T} h_E |[\sigma_h \nu_E]|^2 \, \mathrm{d}s, \tag{1.1}$$

for one element T (of diameter h_T) with edges E (of length h_E) on its boundary ∂T; f is a given volume force and $\mathrm{div}_{\mathcal{T}}\sigma_h$ is the piecewise divergence while $[\sigma_h \nu_E]$ denotes the jump of the normal tractions $\sigma_h \nu_E$ across the element edge E in the direction ν_E (and standard modification on parts of the boundary of Ω with applied surface loads). The estimator of [25] presents, however, also other terms related to the plastic region where the functional analytical setting required for perfect plasticity provides only very weak approximation properties of the displacement field in BD(Ω) [40,20]. It is shown in [4,11] that (1.1) yields a reliable and efficient error estimator

$$\eta_{\mathrm{R}} = \left(\sum_{T \in \mathcal{T}} \eta_T^2 \right)^{1/2}, \tag{1.2}$$

that is, there exist constants $C_{\mathrm{R,eff}}$ and $C_{\mathrm{R,rel}}$ and higher order terms (hot) such that the following bounds hold:

$$C_{\mathrm{R,eff}}\eta_{\mathrm{R}} \leqslant \|\sigma - \sigma_h\|_{L^2(\Omega;\mathbb{R}^{d\times d})} + \mathrm{hot}, \quad \|\sigma - \sigma_h\|_{L^2(\Omega;\mathbb{R}^{d\times d})} \leqslant C_{\mathrm{R,rel}}\eta_{\mathrm{R}} + \mathrm{hot}. \tag{1.3}$$

The estimates $(1.3)_1$ and $(1.3)_2$ are next referred to as efficiency and reliability estimate, respectively. The constants $C_{\mathrm{R,eff}}$ and $C_{\mathrm{R,rel}}$ depend neither on the mesh-sizes h_T, h_E nor on the unknown exact solution [14]; they depend on the domain, the material law and material (hardening and viscosity) parameters. For more general error norms (or error functionals) the duality approach of [37,38,33,17] could be employed. The residuals of the computed solution are multiplied by local weights obtained numerically from the solution of linearized dual problems. Although the theoretical justification of these estimates is disputable, this approach leads to very accurate error guesses and is very valuable for particular error functionals.

In this paper, we consider L^2 stress error control in elastoplasticity and viscoplasticity with hardening [11,4,12,15]. Therein, in each time step, the functional analytical context of linear elasticity is applicable on the price of that some constants crucially depend on the hardening moduli. Averaging [12], explicit residual type [25,11,4], and heuristic estimates [8,36] of the error have been considered and analysed for the finite element error control of incremental plasticity. In these contributions, the above dependence has been always mentioned but never investigated. For explicit residual-based a posteriori error estimators it is moreover known for elliptic problems [13] that the strict estimation of $C_{\mathrm{R,rel}}$ often leads to a huge overestimation and hence appears almost useless as stopping criterion. For sharp error control, on contrary, the implicit error estimators such as the equilibration error estimator [28,2] have proved to give a better performance [13]. This is also confirmed by several applications of the equilibration techniques for the construction of admissible solutions used in the family of error measures based on the error in the constitutive equations for associative material models [21,30].

The main goals of this work are therefore the following. First, we want to explore in an explicit way the type of dependence of the reliability constants on the material properties, in particular on the hardening modulus and the viscosity coefficient for a quite general class of inelastic material models. Instrumental are the establishment of some bounds on the plastic strain error in terms of the stress error, useful also in a broader context than error control of finite element approximations. Second, we want to analyse the feasibility of equilibration technique for efficient and reliable error control of the space discretization error. Finally, we want to investigate the numerical performance of adaptive finite element schemes with marking strategy based on the max-refinement rule and on the local equilibrated error estimators $\eta_{\mathrm{EQ},T}$.