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Characteristic boundary conditions for the two-step Taylor–Galerkin FEM

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Abstract

A general framework for implementing numerical boundary conditions, based upon the rigorous application of characteristic theory, has been developed for the two-step Taylor–Galerkin FEM scheme. The method consists of solving the compatibility equations for the temporal change in characteristic variables using the first step of the two-step Taylor–Galerkin FEM. This application of characteristic boundary conditions is consistent with the spatial and temporal discretization of the two-step Taylor–Galerkin FEM. It is ideal for domains discretized with linear unstructured finite elements as time and space extrapolation from interior elements is not required. Boundary conditions are constructed from the characteristic solution for solid wall and symmetry boundaries, flow exits, and flow inlets. Two simulations are shown highlighting the performance of these boundary conditions.
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Keywords: Two-step Taylor–Galerkin; Linear unstructured finite elements; Characteristic boundary conditions; Compatibility equations; Compressible flow

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1. Introduction

Boundary conditions are an integral part of the solution to partial differential equations. All too often when a solution method is discussed in the literature, the topic of boundary conditions is glossed over or completely ignored. However, the proper treatment of boundary conditions for a particular scheme is as important as the scheme itself. The governing equations of compressible inviscid flow are a system of first-order hyperbolic partial differential equations that constitutes an initial-boundary value problem. The initial conditions must satisfy the governing equations at some reference time level, generally at the

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beginning of the simulation. For finite domains, the boundary conditions supply the domain with information, accounting for physical effects external to the domain, required to achieve a unique solution. In other words, a set of initial conditions and boundary conditions that satisfy the governing partial differential equations establish a well-posed mathematical problem.

For the Euler equation system (hyperbolic wave propagation-dominated), obtaining a set of boundary conditions requires a characteristic analysis of the governing equations to determine which variables must be specified at a boundary for a given boundary condition type and which variables must be computed. It is the unspecified boundary conditions that need to be computed from a discretization that is consistent with the order of accuracy and stability criteria of the solution scheme for the interior of the domain. This paper focuses on the development of a new method of applying characteristic theory for determining the unspecified boundary conditions in compressible flows. Specifically, this new method is designed for computing boundary conditions in the convective flux boundary integral of the second step of the explicit two-step Taylor–Galerkin FEM scheme [3] incorporating linear unstructured finite elements [6].

The two-step Taylor–Galerkin FEM scheme [6] is a combination of the two-step Lax–Wendroff [8] scheme with the standard Galerkin FEM. The first step of the two-step Taylor–Galerkin FEM scheme is an advective predictor designed to advance the solution to the half-step, yielding elemental solution values. The second step incorporates these elemental values into the convective flux integral. This volume integral must be integrated by parts to avoid differentiability constraints on elemental values of the convective flux. Integration by parts introduces a boundary integral for which flow variable information is required. The recommended method to determine the flow variables on the boundary for the two-step Taylor–Galerkin FEM scheme [9] is to follow Usab and Murman's [10] approach. In their approach, a linearized set of characteristic variables are cast in predictor–corrector form. On the boundaries, the predictor step consists of summing the element contributions common to a boundary node. The corrector step consists of enforcing the specified flow variables on the boundary using a simple wave type treatment. This boundary condition approach appears to work well with external flows where flow inlets and exits are an extended distance away from internal boundaries, as in the case of transonic wing simulations. However, we have found that for internal flows with strong interactions with flow boundaries (i.e. shocks) that this approach gives less than desirable results. We have found that this is especially true for transient simulations where unstable results can occur.

We describe here an application of numerical boundary conditions derived from characteristic theory and based upon the wave-like solutions for the convective components of the governing hydrodynamic equations. Thus, only the inviscid form of the equations governing compressible fluid flow are considered for this research. Viscous terms are elliptic in nature and are generally addressed with no-slip or gradient type boundary conditions. With the inviscid form, each wave-like solution can be associated with a characteristic surface that propagates in a characteristic direction with a characteristic velocity. The characteristic directions are determined by rotating the flux vector Jacobian of the homogeneous part of the governing equations into wave propagation directions. The rotation operation results in a set of compatibility equations that are a linear combination of the original governing hydrodynamic equations. In simpler terms, the rotation transforms the governing equations from Cartesian coordinates to wave-propagation coordinates. The compatibility equations are discretized in nearly identical fashion as the first step of the two-step Taylor–Galerkin FEM scheme. This boundary discretization approach is consistent with the spatial and temporal accuracy of the two-step Taylor–Galerkin FEM. The solution of the compatibility equations are combined with the known, or specified, boundary information to yield a complete set of flow variables at the boundary for the convective flux boundary integral of the second step of the two-step Taylor–Galerkin FEM.

This paper provides a detailed development of this new approach for applying characteristic theory to generate boundary conditions for the explicit two-step Taylor–Galerkin FEM. A brief description of the governing hydrodynamic equations of inviscid compressible flow, the corresponding compatibility