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Non recursive Nonlinear Least Squares for periodic signal fitting

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A non-recursive version of Nonlinear Least Squares Fitting for frequency estimation is presented. This problem yields a closed-form solution exploiting a Taylor's series expansion. Respecting some conditions, the computational complexity is reduced, but equally the method assures that the accuracy reaches the Cramer-Rao Bound. The proposed method requires a frequency pre-estimate. A series of simulations has been made to determine how accurate the pre-estimate should be in order to ensure the achievement of the Cramer-Rao Bound in various conditions for different periodic signals. The execution time of the proposed algorithm is smaller compared to a single iteration cycle of the standard approach. The proposed method is useful in applications that require a high accuracy fitting of periodic signals, especially when limited computational resources are available or a real-time evaluation is needed.

1. Introduction

The fitting of a sampled data record of a periodic signal is an important problem typical of many applications. The most studied approaches are based on least squares fitting. In literature two least squares approaches can be found. One is based on a single sinusoid model (sine fitting), the other uses a multi-harmonic model (multi-harmonic fitting). When the recorded periodic signal is not a pure tone, the sine fitting produces a biased estimate due to the presence of harmonics [1]. If the fundamental frequency is known, the least-square procedure can be solved as linear system; in this case, stable and efficient methods are available [2]. On the other hand, if the frequency is unknown, the problem has not a closed-form solution. In this case, one needs to use an optimization algorithm in order to iteratively search the solution [2]. The statistical properties of these methods are largely discussed in literature [1–4].

This paper is focused on the fitting of periodic signals with unknown frequency. In literature, two main approaches are proposed.

The first is based on the transformation of the nonlinear least-squares estimation into a recursive linear least-squares procedure. For each iteration, this method solves a linear system using the frequency value resulting from the previous iteration. The solution of the linear system provides each step the estimate of the harmonics level and a new adjusted frequency. The process runs until a minimum threshold of the mean square error (MSE) is reached. This method is proposed and standardized in the IEEE Standard 1057 [5]. Many methods have been introduced either to reduce the computational demand required to solve the linear system [2,6,7], or to optimize the multidimensional searching [8–11].

The growing demand of portable instrumentation has increased the interest in the efficient implementation of this class of algorithms[12]. In [13,14] methods to reduce the computational load and the memory demand of this least-squares estimators have been proposed.

The second general approach is based on the separation of the nonlinear and linear problems [2,15,16]. The solution of the former provides the frequency estimate which allows to find the other parameters simply solving a linear system.

Focusing on the nonlinear problem, it is possible to write an expression for the frequency estimate depending only on the frequency using the projection matrix. The frequency estimate can be then obtained through the maximization of this expression. This approach requires a one-dimensional iterative search. This method is called Nonlinear Least Squares (NLS) [2]. Moreover, the maximization process is nontrivial, and as such, represents a limit for the applicability of NLS.

In this work, the NLS method is studied with the aim to propose a non-recursive Nonlinear Least Squares (nrNLS), to overcome the above difficulty. The proposed nrNLS approximates the NLS function providing a frequency estimate in a closed-form, thus avoiding the recursion. Great attention has been paid to the determination of the conditions that assure an estimate accuracy near to the Cramer-Rao Bound (CRB).

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It is important to remark that there are a wide range of frequency estimators that can be used as an alternative to NLS [17], based on the interpolation of the peak of the Discrete Fourier Transform (ipDFT), on the periodogram maximization, or zero crossing [18,19]. These methods have the advantage of being faster than NLS. However, when only few cycles of the observed signals are available, they cannot ensure high accuracy. In these cases, it is often necessary to use methods based on least squares [15], as the NLS.

The core of the proposed method is based on the development of the derivative of the NLS function in Taylor's series around a pre-estimate, in order to reduce the frequency estimation problem to a calculation of polynomial roots.

In previous works [20,21], a non-recursive technique for the estimation of the grid frequency based on a similar approach has been proposed. In this work, a more general approach is presented. The new algorithm does not require any filtering, in contrast with [21]. Moreover, the new matricial formulation improves the usability of the algorithm for hardware implementation in different fields of application.

In Section 2 the classical NLS approach is shortly presented, while in Section 3 the proposed nrNLS is derived. The cases of single tone and the multi-harmonic models are separately treated. In Section 4, some results are showed with the aim to evaluate the performances of the proposed method and to understand its validity limit. In Section 5 an example of application of the presented algorithm to the grid voltage analysis is showed.

2. Nonlinear Least Squares (NLS) fitting

The accurate estimation of the frequency and the harmonic content of a sampled periodic signal can be performed using the multi-harmonic least squares fitting procedure. This approach is a parametric method that produces an asymptotically unbiased estimate of the parameters [2]. Moreover, especially when the data record covers only few cycles or a portion of cycle of the signal under analysis, NLS overcomes the well-known problems afflicting the approaches based on the Periodogram maximization or on the Discrete Fourier Transform (DFT) [15].

The generic least squares fitting procedure is based on the minimization of the sum of the squares of the residuals shown in (1).

$$\sum_{n=0}^{N}(\boldsymbol{y}_n - M_n[\vartheta])^2 \tag{1}$$

The residuals are defined as the differences between the measured data and the values assumed by a specific parametric model at the same instants. In (1), $\boldsymbol{y_n} = [y_0 y_1 y_2 \dots y_N]$ is the vector containing the samples of the measured signal taken at instants $\boldsymbol{t_n} = [t_1 t_2 \dots t_N]$ and $\boldsymbol{M_n}[\vartheta]$ is the vector containing the values that the adopted model (depending on the parameters ϑ) assumes at instants $\boldsymbol{t_n} = [t_1 t_2 \dots t_N]$.

In case of multi-harmonic model, $\boldsymbol{M_n}[\vartheta]$ is defined as in (2).

$$\boldsymbol{M_n}[\vartheta] = C + \sum_{h=1}^{H}(a_h \cos(h\omega t_n) + b_h \sin(h\omega t_n)) \tag{2}$$

If H harmonics are included in the model, the number of parameters to estimate are 2H + 2 and listed in the parameters vector ϑ:

$$\vartheta = [a_1 b_1 a_2 b_2 \dots a_H b_H C\omega] \tag{3}$$

The frequency ω makes the problem nonlinear. This is the reason why a closed-form solution of this least-square fitting does not exist. The problem can be split in two stages. In the first one, the estimate $\hat{\omega}$ can be obtained by a 1-D searching, maximizing the expression shown in (4):

$$\hat{\omega} = \max_{\omega}[\mathbf{y}^{\mathrm{T}}\mathbf{D}(\omega)(\mathbf{D}(\omega)^{\mathrm{T}}\mathbf{D}(\omega))^{-1}\mathbf{D}(\omega)^{\mathrm{T}}\mathbf{y}] \tag{4}$$

where the observation matrix $\mathbf{D}(\omega)$ is a $\mathrm{N} \times (2\mathrm{H}+1)$ matrix defined as in (5).

$$\mathbf{D}(\omega) = \begin{bmatrix} \cos(\omega t_1) & \sin(\omega t_1) & \cos(2\omega t_1) & \sin(2\omega t_1) & \dots & \cos(H\omega t_1) & \sin(H\omega t_1) & 1 \\ \cos(\omega t_2) & \sin(\omega t_2) & \cos(2\omega t_2) & \sin(2\omega t_2) & \dots & \cos(H\omega t_2) & \sin(H\omega t_2) & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \cos(\omega t_N) & \sin(\omega t_N) & \cos(2\omega t_N) & \sin(2\omega t_N) & \dots & \cos(H\omega t_N) & \sin(H\omega t_N) & 1 \end{bmatrix} \tag{5}$$

This kind of approach is called Nonlinear Least Squares (NLS).

The second one consists in the solution of the trivial linear system:

$$\hat{\boldsymbol{\theta}} = (\mathbf{D}(\hat{\omega})^T\mathbf{D}(\hat{\omega}))^{-1}\mathbf{D}(\hat{\omega})^T\boldsymbol{y} \tag{6}$$

using $\hat{\omega}$ found in the first stage. The vector $\hat{\boldsymbol{\theta}} = [\hat{a}_1 \hat{b}_1 \hat{a}_2 \hat{b}_2 \dots \hat{a}_H \hat{b}_H \widehat{C}]$ contains the estimates of all other parameters except ω.

3. The proposed method

In this section the proposed method of frequency estimation concerning the first stage of the periodic signal fitting is introduced. It uses an approximation of the NLS approach to estimate the frequency ω_a of a noisy periodic signal. The function to maximize $\mathrm{J}(\omega)$, which depends only on the variable ω, can be written as

$$\mathrm{J}(\omega) = \mathbf{y}^{\mathrm{T}}\mathbf{D}(\omega)(\mathbf{D}(\omega)^{\mathrm{T}}\mathbf{D}(\omega))^{-1}\mathbf{D}(\omega)^{\mathrm{T}}\mathbf{y} \tag{7}$$

where $\mathbf{y}$ is the vector containing the signal samples. The estimate $\hat{\omega}_a$ can be written as

$$\hat{\omega}_a = \max_{\omega}[\mathrm{J}(\omega)] = \max_{\omega}[\mathbf{\Gamma\Psi\Gamma}^T] \tag{8}$$

calling $\mathbf{\Gamma} = \mathbf{y}^{\mathrm{T}}\mathbf{D}$ and $\mathbf{\Psi} = (\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}$ and omitting the dependencies on ω.Note that $\mathbf{\Gamma}$ is a $1 \times (2\mathrm{H}+1)$ vector and $\mathbf{\Psi}$ is a $(2\mathrm{H}+1) \times (2\mathrm{H}+1)$ square matrix.

The value $\hat{\omega}_a$, that maximizes the function (7), also nullifies the derivative with respect to ω of $\mathrm{J}(\omega)$:

$$\mathrm{J}_{\omega}(\omega) = \frac{d\mathrm{J}(\omega)}{d\omega} = 2\mathbf{\Gamma}_{\omega}\mathbf{\Psi}\mathbf{\Gamma}^{\mathrm{T}} + \mathbf{\Gamma}\mathbf{\Psi}_{\omega}\mathbf{\Gamma}^{\mathrm{T}} \tag{9}$$

where with the operator $[\cdot]_{\omega}$ is indicated the elementwise derivative with respect to ω. In the above expressions, it has been exploited the symmetry of the matrixes $\mathbf{\Psi}$, that implies $\mathbf{\Gamma}_{\omega}\mathbf{\Psi}\mathbf{\Gamma}^{\mathrm{T}} = \mathbf{\Gamma}\mathbf{\Psi}\mathbf{\Gamma}_{\omega}^{\mathrm{T}}$. It is important to highlight again that $\mathrm{J}(\omega)$ and its derivative $\mathrm{J}_{\omega}(\omega)$ are function only of the variable ω. $\mathrm{J}_{\omega}(\omega)$ is a transcendental equation and a closed-form expression of its roots does not exist. This is the reason why an iterative method is still required. In Fig. 1 the function $\mathrm{J}(\omega)$ (top) and its derivative $\mathrm{J}_{\omega}(\omega)$ (bottom) of a generic sinusoidal function with additive Gaussian noise are shown.

The basic idea of the proposed approach to estimate ω_a consists in the expansion of $\mathrm{J}_{\omega}(\omega)$ in Taylor series of g order around a central frequency ω_0, as shown in (10).

$$\mathrm{J}_{\omega}(\omega) \simeq \sum_{i=0}^{g}\frac{1}{i!}\frac{d^i\mathrm{J}_{\omega}(\omega)}{d\omega^i}\bigg|_{\omega=\omega_0}(\omega-\omega_0)^i = \sum_{i=0}^{g}\frac{1}{i!}\mathrm{J}_{\omega}^{(i)}(\omega_0)\Delta\omega^i \tag{10}$$

In Eq. (10) $\Delta\omega$ indicates the difference between ω and the central frequency ω_0. Fig. 2 shows as the Taylor's polynomials is a good approximation of the function $\mathrm{J}_{\omega}(\omega)$ around the central frequency ω_0.

If ω_0 is sufficiently close to the actual frequency ω_a, the estimate $\hat{\omega}_a$ can be calculated as a zero of polynomial (10) instead of the zero of the function $\mathrm{J}_{\omega}(\omega)$. In this way, the estimation is no longer recursive since the analytic expressions of the roots of a polynomial up to the fourth order are well known. Unlike the clas-