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Generalization of the Poincare sphere to process 2D displacement signals

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A R T I C L E I N F O

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A B S T R A C T

Traditionally the multiple phase method has been considered as an essential tool for phase information recovery. The in-quadrature phase method that theoretically is an alternative pathway to achieve the same goal failed in actual applications. The authors in a previous paper dealing with 1D signals have shown that properly implemented the in-quadrature method yields phase values with the same accuracy than the multiple phase method. The present paper extends the methodology developed in 1D to 2D. This extension is not a straight forward process and requires the introduction of a number of additional concepts and developments. The concept of monogenic function provides the necessary tools required for the extension process. The monogenic function has a graphic representation through the Poincare sphere familiar in the field of Photoelasticity and through the developments introduced in this paper connected to the analysis of displacement fringe patterns. The paper is illustrated with examples of application that show that multiple phases method and the in-quadrature are two aspects of the same basic theoretical model.

1. Introduction

In a previous paper [1], the authors developed a 1D model of fringe patterns analysis based on the general Theory of Signal Analysis. The current paper extends the 1D model to 2D. The extension to a higher dimension reviews basic concepts of image analysis and adds additional derivations to the subject matter of [1]. These derivations are needed to extend the conclusions arrived in [1] to the case of 2-D. The extension from 1D to 2D is not trivial and this paper will be limited to 2-D images that represent 2D displacement fields. The extension of the derivations of 2D to 3D again is not straightforward and cannot be covered in a single journal paper. The paper will focus on the process of information extraction. Image information is recorded as levels of gray that should be converted into data providing displacement fields and displacement derivatives in the case of deformed bodies.

In the developments of fringe pattern analysis, the signals in quadrature technique associated with the Hilbert transform was postulated as a procedure to get phase [2]. In the current literature, the signals in quadrature method is considered as a symbolic procedure but not a practical tool to get phase. Currently, phase retrieval is based on the multiple phase method and there is an extensive literature on this subject including the challenging extension to dynamic cases.

In [1], it was shown that the Hilbert transform provides phase information with the same accuracy that the multiple phase method and both methods are two possible approaches based on the same basic theory of fringe pattern analysis. Since the Hilbert transform is applicable to one dimension, the extension to 2-D requires additional theoretical developments. In this paper it will be shown that the signals in quadrature technique presented in [1] for 1D signals can be extended to 2D signals. The procedure can be applied to any type of fringe pattern whether it contains cluster of closed fringes or not. Hence, fringe pattern information can be retrieved from one single image recorded under the more general conditions.

2. Two dimensional signals

Fig. 1a illustrates a 2D sinusoidal signal. It has an amplitude and period p, as is the case in 1D. In 2D, one additional degree of freedom is present, local orientation. There are two possible ways to define the orientation: one can choose the direction corresponding to the locus of equal intensity (yellow line of Fig. 1) and the angle this line makes with the x-axis or by the normal **n** in Fig. 1 and the angle θ that the normal **n** makes with the axis x. The yellow line shows a line of equal intensity (phase) while the vector **r** identifies a point P of phase ϕ in the uniform field of the 2D sinusoidal signal.

In [1], it is shown that data analysis in one dimension requires the description of gray levels in terms of 2D complex functions (analytical functions), that leads to the introduction of the concept of phasor:

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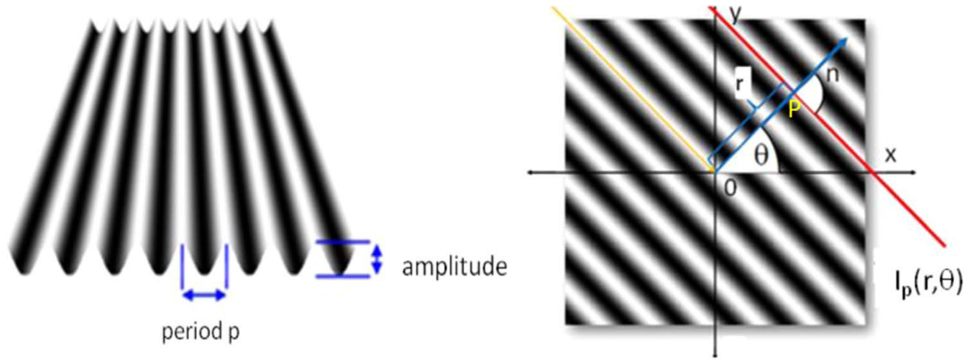

Fig. 1. Sinusoidal 2D signal. I_p intensity at a point of coordinates (**r**, θ). (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

$$\overrightarrow{I_{sp}}(x) = I_{sp}(x)e^{2\pi j\phi(x)} \tag{1}$$

The symbol "⇒" indicates a vector in the complex plane. A phasor in the complex plane is characterized by two separate pieces of information: an amplitude related to the light intensity at the considered point and a phase representing the optical path followed by the recorded wave front from a selected reference point where the phase is assumed to be zero. The classical definition of phase in optics is,

$$\phi(x) = \frac{2\pi\delta(x)}{p} \tag{2}$$

where δ(x) is the optical path and p is the pitch of the sinusoidal function, unit of measure utilized to evaluate a path length and converting distances into angles. Then, along the normal **n**, Eq. (1) becomes,

$$\overrightarrow{I_{sp}}(r, \theta) = I_{sp}(r, \theta)e^{2\pi j\phi(r,\theta)} \tag{3}$$

In 2D, the phase becomes a function of two independent variables, distance **r** and fringe orientation θ. According to the schematic of Fig. 2 and reported nomenclature, the carrier has the equation,

$$I(r, \theta) = I_0 + I_1 \cos\psi(r, \theta) \tag{4}$$

where,

$$\psi(r, \theta) = 2\pi f_c r(\theta) + \alpha \tag{5}$$

Fig. 3 illustrates the carrier signal along a given line. The initial signal is a sinusoidal carrier but after modulation it is transformed into a modulated frequency signal that will include multiple harmonics.

As it is shown in Fig. 4, by filtering it is possible to separate the background from the signal. Each harmonic has its own phase and its own amplitude, and the signal is the sum of many phasors.

In the literature of isothetic lines (moiré fringes), the fringes are represented by Eq. (6),

$$I(x) = I_0 + I_1 \cos\phi(x) \tag{6}$$

where the background amplitude is a constant I_o, the first harmonic is another constant I_1 and the phase ϕ(x) contains the displacement information. This model has been successful in the implementation of the multiple phase method as it has been mentioned but has not worked for the in-quadrature signals before the approach introduced in [1]. In what follows this apparent anomaly will be analyzed.

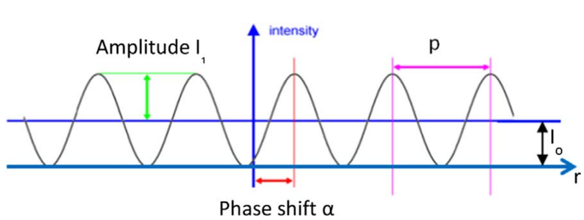

Fig. 2. Original sinusoidal signal.

3. Actual signals and theoretical model signals

Fig. 5 shows a good quality fringe pattern cross-section. This pattern presents the main characteristic of real optical signals extracted from a sensor after a filtering process has smoothed out the quantization effects in space and intensity. The signal is both amplitude and frequency modulated and the background is not a constant. These changes are consequence of the electro-optical devices utilized to obtain the image.

The FFT of such fringe patterns shows [1] that the frequency spectrum contains terms corresponding to the geometrical configuration of the analyzed surface. The FFT includes also the different harmonics corresponding to frequency modulation, not only the first harmonic shown, for example, in Fig. 4 but also higher order harmonics. To these terms are added the terms corresponding to the amplitude modulation of the fringes. All these terms overlap in the FFT and to extract information corresponding to the displacements from the rest of the information it is necessary to apply methods that can accurately separate the different components.

The answer to the question of separating the different components contained in a signal is given by the Bedrosian-Nuttal's theorems [3–5]. A graphical interpretation of the theorems is given the spectrum shown below in Fig. 6. The figure shows the components of the spectrum of a fringe signal whose spectrum satisfies the Bedrosian-Nuttal's theorems. From the spectrum it is possible to conclude that the frequencies of background and amplitude modulation terms must be much smaller than the frequencies of the signal and that the spectra of the harmonics must also be separated, for a more detailed analysis see [1].

There is still one important point to justify, the representation of the optical signal by a single phasor. For that purpose, it is necessary to introduce the concept of analytic signals that is tied with the Hilbert transform concept.

The Hilbert transform is defined by the following expression,

$$I_q(x) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{I_p(\eta)}{x - \eta} d\eta = \frac{1}{\pi}\left(I_p(x) * \frac{1}{x}\right) \tag{7}$$

The Hilbert transform of $I_p(\eta)$, where η is a dummy variable of integration, is the Cauchy principal value indicated in Eq. (7) by PV and can be thought as the convolution of the signal with the function 1/px.

Fig. 7 illustrates the meaning of the Hilbert transform; it is a phase transformation that converts cosine into sine. The bottom part of the figure gives the symbolic representation in the FT. The Hilbert transform takes the original signal, expressed as level of gray or intensity in some scale, and associates the gray level with an analytical function (see Fig. 8):

$$I_{sp}(x) = I_p(x) + jI_q(x) \tag{8}$$

where the symbol "j" denotes the imaginary versor, $I_p(x)$ is the recorded signal (in-phase signal) and $I_q(x)$ is the in-quadrature signal that provides the phase,