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Spline based least squares integration for two-dimensional shape or wavefront reconstruction

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ABSTRACT

In this work, we present a novel method to handle two-dimensional shape or wavefront reconstruction from its slopes. The proposed integration method employs splines to fit the measured slope data with piecewise polynomials and uses the analytical polynomial functions to represent the height changes in a lateral spacing with the pre-determined spline coefficients. The linear least squares method is applied to estimate the height or wavefront as a final result. Numerical simulations verify that the proposed method has less algorithm errors than two other existing methods used for comparison. Especially at the boundaries, the proposed method has better performance. The noise influence is studied by adding white Gaussian noise to the slope data. Experimental data from phase measuring deflectometry are tested to demonstrate the feasibility of the new method in a practical measurement.

1. Introduction

Two-dimensional integration methods [1–4] are widely applied to reconstruct the height or wavefront from the measured gradient data in slope metrology, such as deflectometry [5,6] and wavefront sensing [7,8] etc.

The pioneer studies in two-dimensional integration can be found in wavefront reconstruction since the late 1970s [9–13]. Among these classical studies, Southwell's method [13] received great attention and success because of its good performance and simple implementation with the well-known Southwell geometry. It becomes the representative of the zonal integration methods. However, the integration accuracy is limited since it assumes the height distribution between two sampling points is only quadratic, which is not always true in reality. Based on this observation, an iterative compensation method was proposed to improve the accuracy [14]. By analyzing the Taylor theorem and truncation error, Li et al. proposed a straightforward method with higher order finite difference format [15], which is elegant and outperforms in a comparison [3] as it is more accurate than the traditional Southwell's method, and faster than the iterative method. Recently, Ren et al. proposed an easy implementation of Li's method for incomplete dataset or even in arbitrary domain [16].

In this work, we present a novel spline based least squares method for two-dimensional shape or wavefront reconstruction from slopes in rectangular grids. Benefitted from high accuracy of spline fitting, the reconstruction accuracy can be improved. A comparative study with Southwell's method and Li's method is conducted in this work. The three methods share the same grid geometry (the Southwell geometry), as shown in Fig. 1. One of the beauties of this grid geometry is that the height reconstruction happens exactly at the same locations of slope measurement.

2. Principle

In the proposed method, the zonal relations of the neighboring height values are described as

$$\begin{cases} z_{m,n+1} - z_{m,n} = \sum_{k=0}^{3} \frac{1}{k+1} c_{m,n,k}^{x} \Delta x_{m,n}^{k+1} \\ z_{m+1,n} - z_{m,n} = \sum_{k=0}^{3} \frac{1}{k+1} c_{m,n,k}^{y} \Delta y_{m,n}^{k+1} \end{cases}, \tag{1}$$

where $\Delta x_{m,n} = x_{m,n+1} - x_{m,n}$ and $\Delta y_{m,n} = y_{m+1,n} - y_{m,n}$ are the x- and y-step sizes at matrix location (m, n) as show in Fig. 1. $c_{m,n,k}^{x}$ and $c_{m,n,k}^{y}$ are the coefficients of the kth order piecewise polynomials starting at (m, n), which are determined through the cubic spline fitting of the mth row of x-slopes and the nth column of y-slopes, respectively.

For instance, the x-slope at $(m, n+1)$ and y-slope at $(m+1, n)$ can be

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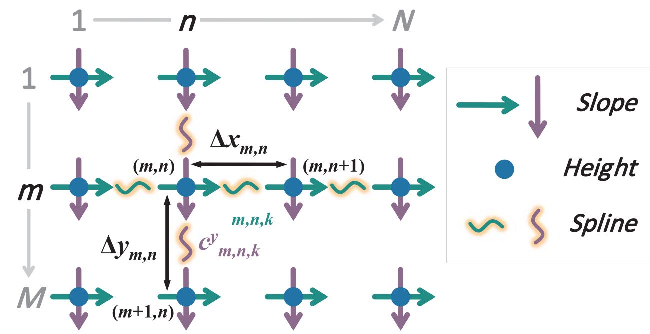

Fig. 1. In Southwell geometry, the height is reconstructed at the same locations where slopes are measured. In our method, slopes are fitted with cubic splines to represent height differences between spacing.

represented by piecewise polynomials starting at (m, n) as

$$s^x_{m,n+1} = \sum_{k=0}^{3} c^x_{m,n,k}\,\Delta x^k_{m,n}. \tag{2}$$

$$s^y_{m+1,n} = \sum_{k=0}^{3} c^y_{m,n,k}\,\Delta y^k_{m,n}. \tag{3}$$

The measured slopes and unknown height are consequently described with piecewise cubic and quartic polynomials, respectively. More significantly, slopes at boundaries of the dataset can be easily and accurately represented by setting the boundary condition of splines as the "natural boundary condition".

By integrating the analytical polynomial functions in Eq. (2) or Eq. (3) with the spline determined coefficients $c^x_{m,n,k}$ or $c^y_{m,n,k}$, the height difference after a lateral step $\Delta x_{m,n}$ or $\Delta y_{m,n}$ can be calculated through the right hand sides of Eq. (1). The linear least squares solution of height can be described as

$$\begin{bmatrix} z_{1,1} \\ z_{2,1} \\ \vdots \\ z_{M,N} \end{bmatrix} = (\mathbf{D}^{\mathrm{T}}\mathbf{D})^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{G}, \tag{4}$$

where the symbol $(\cdot)^{\mathrm{T}}$stands for the transpose operation, and $(\cdot)^{-1}$ is the matrix inverse. The sparse matrix $\mathbf{D}$ and vector $\mathbf{G}$ are

$$\mathbf{D} = \begin{bmatrix} -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 & 1 \\ -1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix}. \tag{5}$$

$$\mathbf{G} = \begin{bmatrix} \sum_{k=0}^{3} \frac{1}{k+1} c^x_{k,1,1} \Delta x^{k+1}_{1,1} \\ \sum_{k=0}^{3} \frac{1}{k+1} c^x_{k,2,1} \Delta x^{k+1}_{2,1} \\ \vdots \\ \sum_{k=0}^{3} \frac{1}{k+1} c^x_{k,M,N-1} \Delta x^{k+1}_{M,N-1} \\ \sum_{k=0}^{3} \frac{1}{k+1} c^y_{k,1,1} \Delta y^{k+1}_{1,1} \\ \sum_{k=0}^{3} \frac{1}{k+1} c^y_{k,2,1} \Delta y^{k+1}_{2,1} \\ \vdots \\ \sum_{k=0}^{3} \frac{1}{k+1} c^y_{k,M-1,N} \Delta y^{k+1}_{M-1,N} \end{bmatrix}. \tag{6}$$

3. Simulation

In order to illustrate the excellent performance of the proposed method, a two-dimensional cosine function with varying local frequencies $z = \cos(2\pi x^2/3000)\cdot\cos(2\pi y^2/3000)$ is selected as the Surface Under Test (SUT) to reconstruct as shown in Fig. 2(a). Its corresponding analytically derived x-slope and y-slope are shown in Figs. 2(b) and (c). We set x-unit the same as y-unit and named as "lateral unit," [l. *u.*], and z-unit is symbolled as [*z. u.*]. The in-plane coordinates are sampled as x=1, 2..., 256 [l. *u.*] and y=1, 2 ..., 256 [l. *u.*]. The value of height ranges in ± 1 [*z. u.*], and both x-slope and y-slope range within ± 1 [*z. u.* / l. *u.*], so the Peak-To-Valley (PTV) of the slopes are 2 [*z. u.* / l. *u.*] for the simulated SUT.

Three integration methods (Southwell's method [13], Li's algorithm 1 in Ref. [15], and our spline-based method in this work) are applied to reconstruct height from the slopes in Fig. 2(b-c) for a comparison. All these methods share the same sparse matrix **D**, which has the less memory cost and computing time for the matrix inverse operation comparing to other two algorithms in Ref. [15]. It is a big advantage in handling huge slope datasets. This is one of reasons why we compare these three methods.

The reconstruction errors are illustrated in Fig. 3. It indicates that these zonal methods make larger reconstruction errors in higher frequency regions. For comparison purposes, Southwell's method has the largest reconstruction error with its Root Mean Square (RMS) $=2.6\times10^{-2}$ [*z. u.*] and PTV $=0.19$ [*z. u.*]. Li's reconstruction method ends up with errors of RMS $=5.8\times10^{-3}$ [*z. u.*] and PTV $=0.17$ [*z. u.*] showing significant improvement compared with Southwell's method. Lastly, the proposed method outperforms the others with reconstruction errors of RMS $=9.6\times10^{-4}$ [*z. u.*] and PTV $=0.03$ [*z. u.*] only.

It is obvious that the proposed spline-based method has better estimation at regions with high-frequency variations. More significantly, splines have naturally good performance at dataset boundaries. In contrast, four neighboring slopes in one direction are always required in Li's algorithm 1 in Ref. [15] which cannot be satisfied at

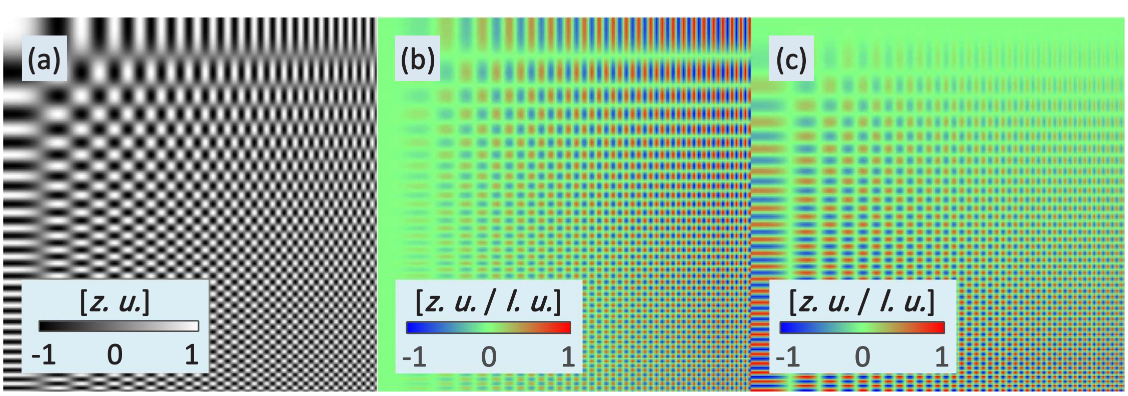

Fig. 2. A surface height (a) with varying local frequencies is chosen as the benchmark in simulation to test the performance of different methods in height reconstruction from x-slope (b), and y-slope(c).