Proceedings of the 36th Chinese Control Conference
July 26-28, 2017, Dalian, China

Consensus of Multi-agent Systems under a Class of Randomly Time-Varying Networks

Kexin Liu[1], Zhisheng Duan[1]

1. State Key Laboratory for Turbulence and Complex Systems, Department of Mechanics and Aerospace Engineering, College of Engineering, Peking University, Beijing 100871, China
E-mail: skxliu@163.com (Liu), duanzs@pku.edu.cn(Duan)

Abstract: Consensus of multi-agent systems under random networks has attracted considerable attention. In previous literatures, it is often assumed that the agents communicate through Markov switching networks. However, in some engineering applications, the available network information is the expectation of the Laplacian matrix and the time-varying networks are not proper to be regarded as Markov switching. To solve this problem, we develop a method to cope with the mean square consensus of multi-agent systems. Based on the properties of expectation of Laplacian matrix, we present the design of relevant parameters and establish some conditions to guarantee the mean square consensus. Numerical simulations are also given to validate our approach.

Key Words: multi-agent systems, randomly time-varying networks, consensus

1 Introduction

As known to all, due to wide application in many fields, such as social sciences, physics, biology, and engineering [1], the consensus of multi-agent systems (MASs) is a central problem of the study of complex networks.

Because of the potential existence of packet dropouts, external disturbances, channel fading, task execution alteration, the communication networks among the agents would be randomly switching topologies [2]. Most of the literatures focused on the Markov switching networks. Just name a few examples. Luan *et al.* considered the finite-time consensus of multi-agent systems (MASs) with stochastic Markov jump topologies and external disturbances [3]. You *et al.* analyzed the mean square consensus of MASs under Markov switching networks [4]. Wang and Zhang considered the distributed output feedback control of of Markov jump multi-agent systems [5]. Savino studied the data-sampled control of MASs with uncertain transition rate [6].

Another kind of random network is with the common assumption that we know the expected value of Laplacian matrix [7–10]. An important application is the link failure analysis of wireless sensor networks. In 2009, Zhou and Wang analyzed the convergence speed in distributed consensus over random switching network by introducing per-step convergence factor [7]. In 2010, Pereira *et al.* considered the mean square consensus of wireless sensor networks with random asymmetric topologies [8]. In 2011, Abaid and Porfiri gave the detailed value of asymptotic mean square convergence factor of numerosity-constrained random networks [9]. Their results were extended to leader-follower consensus over numerosity-constrained random networks in [10].

The above literatures mainly focused on the consensus of MASs with discrete-time dynamics. In addition, the random network links are assumed to be with independent and identical distribution. To our knowledge, there are few literatures investigating the consensus of continuous-time MASs, where the distribution of links are not independent and identical. In this paper, we consider the design of consensus protocols under a more general random distribution network. Conditions are established to ensure mean square consensus according to the time-varying expectation of Laplacian matrix.

The rest of this paper is organized as following. In Sec. 2, the problem description is given. Main results are given in 3. In Sec. 4, simulations are proposed to verify the effectiveness of our design. Sec. 5 concludes our paper .

2 Problem Statement

A directed graph is denoted as $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, 2, \cdots, N\}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. The adjacency matrix G of the graph $\mathcal{G}$ is defined as $G = [g_{ij}]_{N\times N}$, where $g_{ji} = 1$ if and only if $(i, j) \in \mathcal{E}$. In this paper, we consider simple graphs, *i.e.*, $(i, i) \notin \mathcal{E}$ and thus $g_{ii} = 0$. A graph is said to be undirected if $(i, j) \in \mathcal{E}$ implies that $(j, i) \in \mathcal{E}$ for every pair of nodes i and j. A path from node i_1 to node i_l is a sequence of ordered edges satisfying $(i_j, i_{j+1}) \in \mathcal{E}$, $j = 1, 2, \cdots, l-1$. If there is one node in the graph such that there exists a directed path from this node to every other node, then the graph contains a spanning tree. The Lapacian $\mathcal{L}$ is defined as $\mathcal{L} = D - G$, where $D = \mathrm{diag}\{d_1, d_2, \cdots, d_N\}$, $d_i = \sum_{j=1}^{N} g_{ij}$.

The dynamics of the MASs takes form of

$$\dot{x}_i = Ax_i + Bu_i, \tag{1}$$

$x_i \in R^n$ is the system state. The controllers are designed as

$$u_i = K\sum_{j=1}^{N} a_{ij}(t)(x_i - x_j), \tag{2}$$

The agents communicated through a randomly time-varying network $\mathcal{G}(t)$. With this controller, we aims to achieve mean square consensus of MASs (1), which is defined below.

Definition 1 *For any $x_i(0) \in \mathbb{R}^n$, the MASs* (1) *is said to achieve mean square consensus, if $E\|x_i - x_j\|^2 \to 0$ as $t \to \infty$, $i, j = 1, 2, \cdots, N$.*

This work is supported in part by National Natural Science Foundation (NNSF) of China under Grants 61673026, 61528301, in part by China Postdoctoral Foundation under Grant 2016M600011.

To establish our main results, some assumptions are listed below.

Assumption 1 *The sample space of the random variable $\mathcal{L}$ is countable set, i.e., there exists a $s > 0$, such that $\mathcal{L} = \{\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_s\}$, where s can be infinite. There exists a function $\theta_i(\cdot) > 0$ and a constant $d_i > 0$, such that*

$$P(\mathcal{L}(t + \Delta t) = \mathcal{L}_i) = P(\mathcal{L}(t) = \mathcal{L}_i) + \theta_i(\Delta t), \qquad (3)$$

where $\lim_{\Delta t \to 0} |\theta_i(\Delta t)|/(\Delta t) \leq d_i$.

Assumption 2 *The pair (A, B) is stabilizable, and there exists a positive definite matrix P, such that*

$$PA + A^\tau P + \sum_{j=1}^{s} d_j P - PBB^\tau P = -Q. \qquad (4)$$

Lemma 1 ([11]) *For a directed network, all the eigenvalues of $\mathcal{L}$ have nonnegative real parts. Zero is a simple eigenvalue of $\mathcal{L}$ if and only if the corresponding graph $\mathcal{G}$ has a spanning tree.*

Lemma 2 *Consider the matrix $C = I_N \otimes A + B \otimes I_n$, where $A \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{n \times n}$. Its eigenvalues set is $\{\lambda(C) = \lambda(A) + \lambda(B)\}$.*

Proof: Suppose that $\rho \in \lambda(A)$ and $\sigma \in \lambda(B)$. Then, there exist vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^N$, such that

$$Ax = \rho x, \quad By = \sigma y. \qquad (5)$$

Thus, we arrive at

$$(I_N \otimes A + B \otimes I_n)(y \otimes x) = (\rho + \sigma) y \otimes x. \qquad (6)$$

This completes the proof. ■

The definition of average dwell time is given below.

Definition 2 *([12]) For arbitrary $T_2 > T_1 > 0$, let $N_\sigma(T_1, T_2)$ denotes the switch number on time interval (T_1, T_2). If*

$$N_\sigma(T_1, T_2) \leq N_0 + \frac{T_2 - T_1}{\tau_a} \qquad (7)$$

holds for any given $N_0 \geq 0$, $\tau_a > 0$, then the constant τ_a is called average dwell time. In general, we set $N_0 = 0$.

3 Main Results

In this section, we will investigate the consensus of MASs under the random time-varying network. To begin with, an important proposition will be introduced.

Proposition 1 *Consider the randomly time-varying system*

$$\dot{x} = A(t)x(t). \qquad (8)$$

Suppose that $A(t) \in \{A_1, A_2, \cdots, A_s\}$. There exists a function $\theta_i(\cdot) > 0$ and a constant $d_i < 0$, such that

$$P(A(t + \Delta t) = \mathcal{L}_i) \leq P(A(t) = A_i) + \theta_i(\Delta t), \qquad (9)$$

where $\lim_{\Delta t \to 0} |\theta_i(\Delta t)|/(\Delta t) \leq d_i$, $d_i > 0$ is a constant. If the following time-varying system

$$\dot{y} = (\bar{A}(t) + \sum_{j=1}^{s} \frac{d_j}{2} I_n) y, \qquad (10)$$

is asymptotically stable, where $\bar{A}(t) = E(A(t))$, then, system (8) is asymptotically stable in sense of mean square.

Proof: Define

$$X_i(t) = E(x(t)x^{\mathrm{T}}(t)\phi_i(t)), \qquad (11)$$

where

$$\phi_i(t) = \begin{cases} 1, \text{ if } A(t) = A_i, \\ 0, \text{ otherwise.} \end{cases}$$

Since

$$E(x^{\mathrm{T}}(t)x(t)) = \mathrm{tr}(E(x(t)x^{\mathrm{T}}(t))) = \mathrm{tr}(\sum_{i=1}^{N} E(x(t)x^{\mathrm{T}}(t)\phi_i(t))),$$

the stability of $X_i(t)$, $i = 1, 2, \cdots, s$, implies the convergence of $E(x^{\mathrm{T}}(t)x(t))$. From the definition of $X_i(t)$, we have

$$\begin{aligned} & X_i(t + \Delta t) \\ =& x(t + \Delta t)x^{\mathrm{T}}(t + \Delta t)P(A(t + \Delta t) = A_i) \\ =& x(t + \Delta t)x^{\mathrm{T}}(t + \Delta t)(P(A(t) = A_i) + \theta_i(\Delta t)) \\ =& \sum_{j=1}^{s} \Phi_j(t + \Delta t, t)x(t)x^{\mathrm{T}}(t)\Phi_j^{\mathrm{T}}(t + \Delta t, t)P(A(t) = A_j) \\ & \times (P(A(t) = A_i) + \theta_i(\Delta t)). \end{aligned}$$

$\Phi_j(t+\Delta t, t)$ is the state transition matrix of system (8) from the time t to $t + \Delta t$, with the assumption that $A(t) = A_j$. Thus,

$$\Phi_j(t + \Delta t, t) = I + A_j \Delta t + o(\Delta t).$$

It arrives at

$$\begin{aligned} & X_i(t + \Delta t) \\ =& \sum_{j=1}^{s} (I + A_j \Delta t)x(t)x^{\mathrm{T}}(t)(I + A_j \Delta t)^{\mathrm{T}} P(A(t) = A_j) \\ & \times (P(A(t) = A_i) + \theta_i(\Delta t)) + o(\Delta t) \\ =& X_i(t) + (E(A(t))X_i(t) + X_i^{\mathrm{T}}(t)E(A(t)))\Delta t \\ & + \sum_{j=1}^{s} X_j(t)\theta_i(\Delta t) + o(\Delta t). \end{aligned}$$

Let $\Xi_i(t) = \mathrm{Vec}(X_i(t))$. It is clear that $\Xi_i(t) \to 0$ implies that $X_i(t) \to 0$, $t \to \infty$. Thus,

$$\Xi_i(t + \Delta t) = \Xi_i(t) + \mathcal{A}(t)\Xi_i \Delta t + \theta_i(\Delta t) \sum_{j=1}^{s} \Xi_j,$$

where $\mathcal{A}(t) = I_n \otimes \bar{A}(t) + \bar{A}(t) \otimes I_n$. The dynamics of Ξ_i takes form of

$$\dot{\Xi}_i = \mathcal{A}(t)\Xi_i + \lim_{\Delta t \to 0} \frac{\theta_i(\Delta t)}{\Delta t} \sum_{j=1}^{s} \Xi_j \leq \mathcal{A}(t)\Xi_i + d_i \sum_{j=1}^{s} \Xi_j.$$

Letting $\Xi = \sum_{j=1}^{s} \Xi_j$, we have

$$\dot{\Xi} \leq (\mathcal{A}(t) + \sum_{j=1}^{s} d_j I_{n^2})\Xi, \qquad (12)$$

It follows from $\Xi = \sum_{j=1}^{s} \mathrm{Vec}(X_j)$ that the convergence of Ξ implies the stability of X_i, $i = 12, \cdots, s$. Given system

$$\dot{\Pi} = (\mathcal{A}(t) + \sum_{j=1}^{s} d_j I_{n^2})\Pi, \quad \Pi(0) = \Xi(0), \qquad (13)$$