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Universal form of the equations governing membrane deformation under hydrostatic pressure for simpler design of sensors and tunable optical devices

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ABSTRACT

Flexible membranes have applications in liquid filled lenses and pressure sensors. They deform under hydrostatic pressure, thus changing the asphericity of the lens and its focal length. This behavior enables tuning of the lens by changing the pressure of the fluid inside. A universal form of the nonlinear differential equations describing the deformation of a flexible membrane is presented here, showing that their solution is valid for membranes having the same thickness to radius ratio and made of materials having the same flexural rigidity and Poisson ratios. Hence by solving the equations once, a simple scaling allows obtaining a set of solutions that matches these ratios. This should simplify the design of tunable lenses and pressure sensors based on flexible membranes. In addition, approximate analytic solutions are presented in a normalized form.

1. Introduction

Membrane based tunable optical devices such as adaptive optical lenses made of liquid filled membranes, pressure sensors and tunable color filters have recently attracted the attention of the optics community due to their small size, high sensitivity and the relative simplicity for devices manufacturing.

Liquid filled microlenses are achieved by using flexible membranes and applying hydrostatic pressure [1–6], and can be tuned by variations to this pressure for the desired focal length. Membrane based tunable microlens under hydrostatic pressure may even be used for achieving desired asphericity for aberrations compensation [7,8]. Flexible membranes may be used for pressure sensing [9], and even for optical wavelength tuning by changing the spacing of a flexible grating applied on a flexible membrane via the membrane deformation [10]. Due to the extended use of flexible membranes in various fields of optics, we would like to present in this article an easy method of calculating the deformation of a flexible membrane under hydrostatic pressure. The method is based on normalization and scaling of the parameters characteristic of the shape and material of the membrane. Scaling laws in physics are well known [11] to simplify prediction of the behavior of systems when the dimensions are changed. Using this approach, differential equations governing the behavior of physical systems can be written in a universal form such as in liquid crystal dynamic deformations and in transmission line theories [12,13].

In this article a set of universal equations are derived for simulating and predicting the deformation of flexible circular membrane clamped at its edges under hydrostatic pressure. This approach involves less computing and allows solving the equations once, in a way that helps describing the deformation for membranes having the same thickness to radius ratios and made of materials having the same flexural rigidity and Poisson ratios. The approach should be helpful for designing tunable optical devices and pressure sensors based on flexible membranes.

2. Problem definition

The problem under consideration maybe described in Fig. 1. A flexible circular membrane is clamped at its edges under hydrostatic pressure. Deformation of the membrane is described by a set of two nonlinear differential equations [14]:

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=-\frac{1-\nu}{2r}\left(\frac{dw}{dr}\right)^2-\frac{dw}{dr}\frac{d^2w}{dr^2}$$
$$\frac{d^3w}{dr^3}+\frac{1}{r}\frac{d^2w}{dr^2}-\frac{1}{r^2}\frac{dw}{dr}=\frac{12}{d^2}\frac{dw}{dr}\left[\frac{du}{dr}+\nu\frac{u}{r}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2\right]+\frac{pr}{2F} \quad (1)$$

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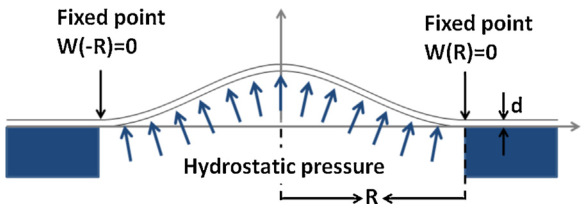

Fig. 1. Schemtic of the geometry of a flexible membrane clamped at its edges under hydrostatic pressure.

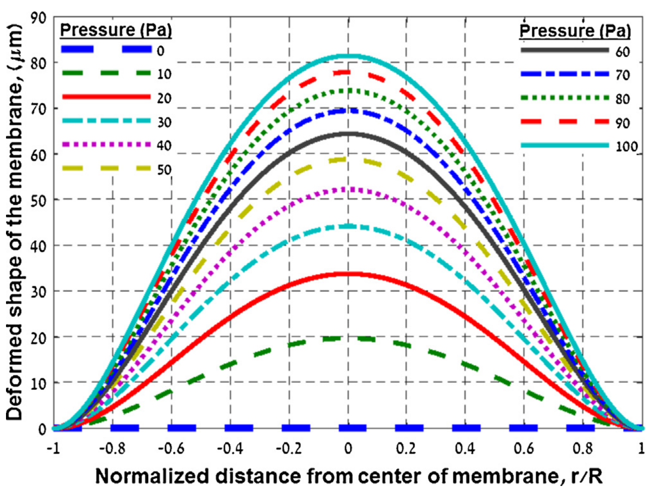

Fig. 2. Deformation of a PDMS membrane (E = 1.12 MPa, ν = 0.48, d = 50 μm) for pressure in the range of 0–100 Pa.

with the boundary conditions:

$$\begin{cases} u = \dfrac{dw}{dr} = 0, & \text{at} \quad r = 0 \\ u = w = \dfrac{dw}{dr} = 0, & \text{at} \quad r = R \end{cases} \tag{2}$$

Here $u(r)$ and $w(r)$ are the displacements in the radial and axial directions r and z respectively, d is the thickness of the membrane, and p is the uniform hydrostatic pressure. The flexural rigidity of the axisymmetric membrane is expressed as: $F = E \cdot d^3/(12(1-\nu^2))$ where E and ν are the Young's modulus and Poisson's ratio, respectively.

In the form presented in Eqs. (1) and (2), one has to insert the thickness and radius of the membrane and the individual parameters of the material it is made of. In what follows we show that it is possible to write these equations in universal form and so with simple scaling the same solution can be used to any membranes having the same thickness to radius ratio and made of material having the same ratio $F' = F/d^3 = E/(12(1-\nu^2))$ which will be addressed hereafter as "material flexural rigidity ratio".

Numerical solution of Eqs. (1) and (2) is presented in Fig. 2 for a polydimethylsiloxane (PDMS) membrane with E = 1.12 MPa, ν = 0.48, thickness d = 50 μm and radius R = 1200 μm.

When dealing with liquid lenses, it is common to describe the deformation of the membrane as an axially symmetric aspherical surface. In this case $w(r)$ can be written as:

$$w(r) = \frac{cr^2}{1 - \sqrt{1-(1+K)c^2r^2}} + \sum_{k=2}^{\infty} a_k r^{2k} \tag{3}$$

where c is the vertex curvature, K is the conic constant of the profile and $c = \frac{\sqrt{8}}{d}C_1, \quad a_k = \frac{\sqrt{2} \cdot C_{2k-1}}{kd^{2k-1}} \quad \{k = 2, 3, \ldots\}$.

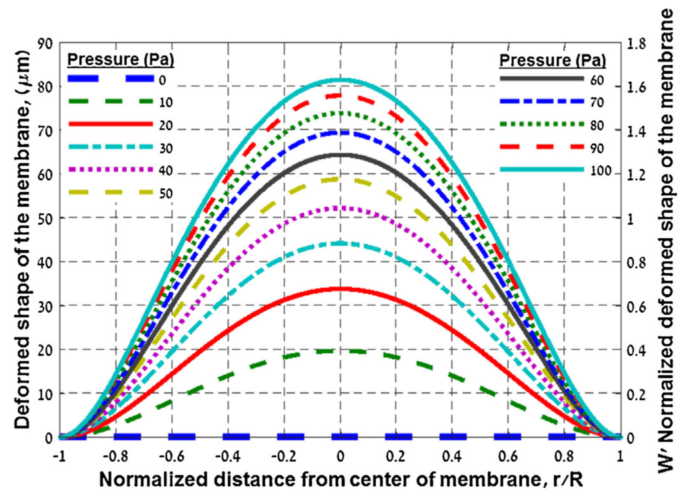

Fig. 3. *Right y-axis*: Solution for the universal equations for membrane deformation (E = 1.12 MPa, ν = 0.48, η = 1/24), $W'(r')$, normalized to the membrane thickness d. *Left y-axis*: Deformation of the membrane after scaling $W'(r')$, by the thickness of the membrane (d = 50 μm) to obtain the deformation in μm for this particular thickness.

This form is helpful when one wishes to find pressure needed to apply for achieving the proper curvature in order to reduce chromatic and spherical aberrations in liquid lenses as presented by Waibel et al. [7,8]. For the geometry presented here, it is safe to assume that the conic constant is $K = -1$.

3. Universal form of the equations

The presented problem can be solved by using a more universal form of the equations that depends on the ratio between the thickness of the membrane to its radius, $\eta = d/R$, on the normalized distance from the middle of the membrane, $r' = r/R$, on the material rigidity ratio $F' = E/(12(1-\nu^2))$ and the Poisson ratio. To obtain this, the following change of variables is used:

$$r' \triangleq \frac{r}{R}, \quad z' \triangleq \frac{z}{d}, \quad u' \triangleq \frac{u}{R}, \quad w' \triangleq \frac{w}{d} \tag{4}$$

The set of the equations then becomes:

$$\begin{aligned} &\frac{d^2u'}{dr'^2} + \frac{1}{r'}\frac{du'}{dr} - \frac{u'}{r'^2} = -\eta^2 \left[\frac{1-\nu}{2r'} \left(\frac{dw'}{dr'} \right)^2 - \frac{dw'}{dr'}\frac{d^2w'}{dr'^2} \right] \\ &\frac{d^3w'}{dr'^3} + \frac{1}{r'}\frac{d^2w'}{dr'^2} - \frac{1}{r'^2}\frac{dw'}{dr'} = \frac{12}{\eta^2}\frac{dw'}{dr'} \left[\frac{du'}{dr'} + \nu\frac{u'}{r'} + \frac{1}{2}\eta^2 \left(\frac{dw'}{dr'} \right)^2 \right] + \frac{pr'}{2\eta^4 F'} \end{aligned} \tag{5}$$

and the boundary conditions are:

$$\begin{cases} u' = \dfrac{dw'}{dr'} = 0, & \text{at} \quad r' = 0 \\ u' = w' = \dfrac{dw'}{dr'} = 0, & \text{at} \quad r' = 1 \end{cases} \tag{6}$$

This problem could be solved easily using *Matlab* built-in function *bvp4c*() for obtaining the numerical solution of Eqs. (5) and (6) that describes the flexible membrane deformation.

The solution of the universal equations for a PDMS membrane with E = 1.12 MPa, ν = 0.48, η = 1/24 and d = 50 μm is presented in Fig. 3. These results are in excellent agreement with the results presented by Choi et al. [15] with the same parameters if we present their results in the normalized form.

Having calculated Fig. 3, now we can simply get the deformation for membranes having the same flexural rigidity and Poisson ratios following scaling the radial coordinate by R and the normalized deformation by the thickness d. Note also that based on Eq. (5), the