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An $\mathcal{H}_2$-type error bound for balancing-related model order reduction of linear systems with Lévy noise

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ABSTRACT

To solve a stochastic linear evolution equation numerically, finite dimensional approximations are commonly used. For a good approximation, one might end up with a sequence of ordinary stochastic linear differential equations of high order. To reduce the high dimension for practical computations, model order reduction is frequently used. Balanced truncation (BT) is a well-known technique from deterministic control theory and it was already extended for controlled linear systems with Lévy noise. Recently, a new ansatz was investigated which provides an alternative way to generalize BT for stochastic systems. There, the question of the existence of an $\mathcal{H}_2$-error bound was asked which we answer in this paper. We discuss how this bound can be computed practically and how to use it to find a suitable reduced order dimension.

1. Introduction

Model order reduction (MOR) is of major importance in the field of deterministic control theory. It is used to save computational time by replacing large scale systems by systems of low order in which the main information of the original system should be captured. Such high dimensional problems occur for example after the spatial discretization of a partial differential equation (PDE) which can be used to model chemical, physical or biological phenomena. A particular ansatz to obtain a reduced order model is to balance a system such that the dominant reachable and observable states are the same. Afterwards, the difficult to observe and difficult to reach states are neglected. One way to do that is to use balanced truncation (BT) which was introduced by Moore [1] and a thorough treatment of the topic can be found in Antoulas [2] or Obinata, Anderson [3].

Since many phenomena in computational sciences and engineering contain uncertainties, it is natural to extend PDE models by adding a noise term. This leads to stochastic PDEs (SPDEs) which are studied in Da Prato, Zabczyk [4] and in Prévôt and Röckner [5] for the Wiener case. Peszat, Zabczyk consider more general equations with Lévy noise in [6], where the solutions may have jumps. To solve SPDEs numerically, one can reduce them to large scale ordinary SDEs by using the Galerkin method. For that reason, generalizing MOR techniques to stochastic systems can be motivated. The mentioned Galerkin approximation is for example investigated in Grecksch, Kloeden [7], Hausenblas [8], Jentzen, Kloeden [9] and Redmann, Benner [10].

To reduce large scale SDEs, balancing related methods are generalized. BT is considered for SDEs with Wiener noise in Benner, Damm [11] and for systems with Lévy noise it is done by Benner, Redmann in [12]. Benner and Redmann provide an $\mathcal{H}_2$-type error bound and the preservation of mean square asymptotic stability is shown in Benner et al. [13]. In Benner et al. [14] and Damm, Benner [15] an example is presented which clarifies that the $\mathcal{H}_\infty$-error bound from the deterministic case does not hold for stochastic systems. Recently, a new ansatz to extend BT to SDEs is considered by Benner et al. [14] or Damm, Benner [15] in which a new reachability Gramian is used. This alternative Gramian so far has no integral representation involving the fundamental solution of the system which is in contrast to the first approach. The advantage of the new ansatz is the existence of an $\mathcal{H}_\infty$-error bound and the preservation of mean square asymptotic stability. It only remains to prove an $\mathcal{H}_2$-error bound to have a closed theory. This $\mathcal{H}_2$-error bound analysis is present in this paper.

In this paper, we focus on BT for SDEs with Lévy noise. We start with giving an overview about the two ways to generalize the deterministic framework and state the most important results that are already proven. In Section 2, we briefly discuss the procedure and emphasize results on error bounds and the stability analysis of the methods. In Section 3, we contribute an $\mathcal{H}_2$-type error bound $\tilde{\epsilon}$ for the new ansatz in [14] and [15] to close the gap in the error bound analysis. The non-negative number $\tilde{\epsilon}$ bounds the worst case

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mean error between the original and the reduced order output Y and Y_R as follows:

$$\sup_{t\in[0,T]} \mathbb{E}\left\|Y(t)-\tilde{Y}_R(t)\right\|_2 \leq \tilde{\epsilon}\, \|u\|_{L^2_T}.$$

As a first step, we provide a representation of $\tilde{\epsilon}$ which can be taken for practical computations and hence be used for finding a suitable reduced order dimension. For this representation, we need to solve three matrix equations which are much cheaper than computing the expected value $\mathbb{E}\left\|Y(t)-\tilde{Y}_R(t)\right\|_2$. Furthermore, we prove that $\tilde{\epsilon}$ can be rewritten as an expression depending on the truncated Hankel singular values (HSVs) of the system similar to the $\mathcal{H}_\infty$ error bound. This second representation can be used to find a suitable reduced order dimension based on the HSVs and it shows that the error bound is small if the truncated states are unimportant (states corresponding to the small HSVs).

2. Balancing of stochastic systems with Lévy noise

Let $A, N_k \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. For $t \geq 0$ and $X(0) = x_0$ we consider the following linear stochastic system:

$$dX(t) = [AX(t) + Bu(t)]dt + \sum_{k=1}^{q} N_k X(t-)dM_k(t), \tag{1}$$

$$Y(t) = CX(t),$$

where $M_1, \ldots, M_q$ are scalar uncorrelated and square integrable Lévy processes with mean zero defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.[1] In addition, we assume M_k $(k = 1, \ldots, q)$ to be $(\mathcal{F}_t)_{t\geq 0}$-adapted and the increments $M_k(t+h) - M_k(t)$ to be independent of $\mathcal{F}_t$ for $t, h \geq 0$. With L^2_T we denote the space of all $(\mathcal{F}_t)_{t\geq 0}$-adapted stochastic processes v with values in $\mathbb{R}^m$, which are square integrable with respect to $\mathbb{P}\otimes dt$. The norm in L^2_T is given by

$$\|v\|^2_{L^2_T} := \mathbb{E}\int_0^T v^T(t)v(t)dt = \mathbb{E}\int_0^T \|v(t)\|_2^2\, dt,$$

where we define the processes v_1 and v_2 to be equal in L^2_T if they coincide almost surely with respect to $\mathbb{P}\otimes dt$. For the case $T=\infty$, we denote the space by L^2. Further, we assume the control $u \in L^2_T$ for every $T > 0$. The solution of Eq. (1) we denote by $X(t, x_0, u)$ and the corresponding output by $Y(t, x_0, u)$. Moreover, we assume mean square asymptotic stability which is

$$\mathbb{E}\|X(t,x_0,0)\|_2^2 \to 0 \quad \text{for } t\to\infty. \tag{2}$$

Below, we set $q = 1$ and $M := M_1$, $N := N_1$ for simplicity of notation. Any of the following results also holds for general q.

2.1. Type 1 balanced truncation

In type 1 BT the idea is to introduce a generalized fundamental solution to the state Eq. (1) which is a matrix-valued process $(\Phi(t))_{t\geq 0}$ defined by $X(t, x_0, 0) = \Phi(t)x_0$. This can be used to define the Gramians

$$P := \int_0^\infty \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right] ds, \tag{3}$$

$$Q := \int_0^\infty \mathbb{E}\left[\Phi^T(s)C^TC\Phi(s)\right] ds.$$

Following the arguments in Section 3 in [12] we know that the Gramians are solutions of generalized Lyapunov equations:

$$AP + PA^T + NPN^Tc = -BB^T, \tag{4}$$

$$A^TQ + QA + N^TQNc = -C^TC, \tag{5}$$

[1] We assume that $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and that $\mathcal{F}_0$ contains all $\mathbb{P}$ null sets.

where $c := \mathbb{E}\left[M(1)^2\right]$. Below, we suppose to have a completely observable and reachable system (1) in terms of the concepts used in [11] or [12] which implies $P, Q > 0$. By Section 3 in [12], we have the following results:

Proposition 2.1.

(i) *The minimal energy to steer the average state to $x \in \mathbb{R}^n$ is bounded from below as follows:*

$$x^TP^{-1}x \leq \inf_{\substack{u\in L^2_T, T>0,\\ \mathbb{E}[X(T,0,u)]=x}} \|u\|^2_{L^2_T}.$$

(ii) *The energy that is caused by the observation of an initial state $x_0 \in \mathbb{R}^n$ is*

$$\|Y(\cdot, x_0, 0)\|^2_{L^2} = x_0^TQx_0.$$

Due to the energy interpretation in Proposition 2.1, we consider the state x to be difficult to reach if the expression $x^TP^{-1}x$ is large and we call it difficult to observe if the term x^TQx is small. If the system is balanced, i.e. $P = Q = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$, then it is ensured that the sets of difficult to observe and difficult to reach states coincide. We can always balance the system as follows. We apply a state space transformation, which does not change the output, by using an invertible matrix T:

$$(A, B, C, N) \mapsto \left(TAT^{-1}, TB, CT^{-1}, TNT^{-1}\right)$$

which leads to transformed Gramians

$$(P, Q) \mapsto \left(TPT^T, T^{-T}QT^{-1}\right).$$

In order to obtain a balanced realization, we choose $T = \Sigma^{\frac{1}{2}}K^TU^{-1}$ with $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n) > 0$. U comes from the Cholesky decomposition of $P = UU^T$ and K is an orthogonal matrix corresponding to the singular value decomposition $U^TQU = K\Sigma^2K^T$. This yields

$$TPT^T = T^{-T}QT^{-1} = \operatorname{diag}(\sigma_1, \ldots, \sigma_n) > 0.$$

The HSVs $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ characterize the importance of a state, the smaller σ_i the more difficult to reach and to observe the corresponding state component. We partition as follows:

$$T = \begin{bmatrix} W^T \\ T_2^T \end{bmatrix} \text{ and } T^{-1} = \begin{bmatrix} V & T_1 \end{bmatrix},$$

where $W^T \in \mathbb{R}^{r\times n}$, $V \in \mathbb{R}^{n\times r}$ and r represents the reduced order model (ROM) state space dimension. Then, the ROM coefficients, obtained by truncation are

$$(A_{11}, B_1, C_1, N_{11}) = \left(W^TAV, W^TB, CV, W^TNV\right).$$

Type 1 balanced truncation preserves mean square asymptotic stability as shown in Theorem 2.3 in [13].

Theorem 2.2. *Let $\sigma_r \neq \sigma_{r+1}$, then for $t \geq 0$ and $X_R(0) = x_{R,0}$ the ROM*

$$dX_R(t) = A_{11}X_R(t)dt + N_{11}X_R(t-)dM(t)$$

is mean square asymptotically stable if for $t \geq 0$ and $X(0) = x_0$

$$dX(t) = AX(t)dt + NX(t-)dM(t)$$

is mean square asymptotically stable.

The result in Theorem 2.2 is essential for the existence of the ROM reachability Gramian $P_R := \int_0^\infty \mathbb{E}\left[\Phi_R(s)B_1B_1^T\Phi_R^T(s)\right] ds$ which occurs in the $\mathcal{H}_2$-type error bound below. Here, Φ_R denotes the fundamental solution of the ROM. The matrix P_R fulfills

$$A_{11}P_R + P_RA_{11}^T + N_{11}P_RN_{11}^Tc = -B_1B_1^T.$$