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Collinear dynamical systems ☆

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ABSTRACT

We introduce and study collinearity as a systems theoretic property, asking whether coupled dynamical systems with collinear initial conditions maintain collinear solutions for all times. Completely characterizing such collinear dynamical systems, we find that they define a Lie algebra whose Lie group are the invertible collinearity-preserving maps. Our characterization of collinear systems then allows us to determine state feedbacks which enforce collinear solutions in coupled control systems. Further, we characterize coupled linear differential equations whose solutions will asymptotically become collinear. Last, we characterize collinearity of multiple dynamical systems and their ability to produce coplanar solutions. Our findings relate to flows on the real projective spaces and Grassmannians.

1. Motivation

Every pair of two points uniquely defines a line that passes through them. If one, in addition, insists that this line shall pass through the origin, then the two points are said to be collinear. It is thus fair to say that collinearity is the simplest nongeneric configuration of two points.

Similarly, every pair of solutions of two (possibly coupled) differential equations uniquely defines a line, varying with time, that passes through the two solutions at every instance of time. Insisting that this line shall pass through the origin at every instance of time, i.e. asking for the two solutions to remain collinear for all times, can thus be seen as the simplest nongeneric configuration of two solutions.

This reasoning alone would justify to study such collinear solutions for conceptual reasons. But it furthermore happens that collinearity is relevant in many technical applications. Since differential equations usually serve as models in such applications, this, moreover, provides a practical motivation for studying collinear solutions.

One application which shall be mentioned in that regard is laser spectroscopy with end-pumped dye lasers. Therein, the pump beam must form a collinear configuration with mirrors and resonator (cf. [2, subsection 5.7.5]). Having actuated mirrors, say piezoelectrically, the goal would be to maintain such a collinear configuration.

Another application in which collinearity plays a crucial role is the design of antenna arrays. Should one aim to increase the power radiated in the plane orthogonal to the array, then one aligns the antennas in the array collinearly (cf. [3, section 8.4]). If the antennas were mobile antennas, then the goal would be to maintain them in such a collinear alignment.

A further motivation to study collinear solutions is to improve our understanding of the dynamical systems producing these solutions.

One case in which collinear solutions indeed helped to understand a dynamical system is the three-body problem, i.e. the Newtonian model of three heavy masses, mutually exerting gravitation on each other. Therein, a family of solutions in which the three bodies remain collinear after having been initialized collinearly, periodically returning to the initial collinear configuration, was determined by Euler in 1767 (cf. [4, subsection 2.3.3]). Together with the equilateral solutions later found by Lagrange, these solutions remain to be fundamental for the understanding of the three-body problem.

Another example in which collinear solutions contributed to a better systems theoretic understanding of a system stems from formation control. Specifically, a research direction sparked by [5] solves formation control problems by constructing scalar fields which are maximal at the desired formations and then applying the gradient of that scalar field as the control action. These scalar fields are constructed in such a fashion that they are regular almost everywhere else, except for certain critical points which only constitute a set of measure zero. These critical points happen to be collinear configurations and it follows that the gradient flow of the proposed scalar field leaves these collinear configurations invariant, i.e., they do not belong to the region of attraction of the desired formation. With that reason, these collinear solutions were studied in detail in [6] in order to understand the convergence properties of the control.

☆ The results in this article are partially contained in a paper that was submitted to the 2017 American Control Conference [1].

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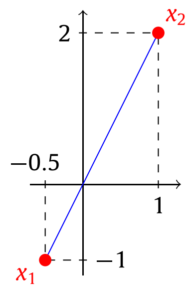

Fig. 1. The depicted two points in $\mathbb{R}^2$ are collinear since they satisfy $x_2 = -2x_1$. Thus, they lie on the same line passing through the origin.

The above examples demonstrate that collinearity of solutions in coupled differential equations shall deserve further attention, for reasons stemming both from technical applications and systems theory. These observations indeed justify to introduce and study collinearity as a systems theoretic property, which is the goal of the present paper (also cf. [1]).

The remainder of this paper is structured as follows: Section 2 comprises the formal problem statement. Thereafter, in Section 3, we characterize collinear dynamical systems. This characterization is exploited for the purpose of finding state feedbacks which render systems collinear in Section 4. In Section 5, we turn our attention to systems that asymptotically become collinear. Last, in Section 6, we study collinearity of multiple systems and coplanarity. Section 7 concludes the paper.

2. Problem statement

Let x_1 and x_2 be two nonzero vectors from $\mathbb{R}^n$. Then one says that x_1 and x_2 are collinear if there is some real scalar α with the property that

$$x_2 = \alpha x_1. \tag{1}$$

This characterization of collinearity just formally says that the line that passes through x_1 and x_2 should also pass through the origin. For instance, let x_1 be $(-0.5, -1)$ and x_2 be $(1, 2)$. Then $\alpha = -2$ scales x_1 in such a fashion that (1) remains satisfied, i.e. x_1 and x_2 are indeed collinear. As we can infer from Fig. 1, in which x_1 and x_2 are depicted, the line that passes through x_1 and x_2 also passes through the origin. If one is interested in vectors x_1 and x_2 that share a line with some fixed nonzero $p \in \mathbb{R}^n$, then, instead of (1), one must consider $x_2 - p = \alpha(x_1 - p)$. In that case, all techniques in the remainder of the paper are still applicable via the affine transformation $x_1 \mapsto x_1 + p, x_2 \mapsto x_2 + p$.

Another way to formulate collinearity is that x_2 shall lie in the span $\{\alpha x_1 | \alpha \in \mathbb{R}\}$ of x_1, or, more geometrically, that the pair (x_1, x_2) shall lie in the vector bundle

$$\bigsqcup_{x_1 \in \mathbb{R}^n} \{\alpha x_1 | \alpha \in \mathbb{R}\} \tag{2}$$

over $\mathbb{R}^n$, within which interpretation x_1 would be seen as a point and x_2 would be seen as a vector. The bundle (2) is depicted left in Fig. 2. Yet another interpretation of collinearity stems from real projective geometry: define the equivalence relation

$$x_1 \sim x_2 \quad :\Leftrightarrow \quad \exists \alpha \in \mathbb{R}: \quad x_2 = \alpha x_1 \tag{3}$$

and recall that the real projective space $\mathbb{RP}^{n-1}$ is the quotient space consisting of equivalence classes of $\sim$. Then collinearity of x_1 and x_2 amounts to their equivalence under $\sim$, i.e. to having them define the same point in $\mathbb{RP}^{n-1}$. As the last equivalent characterization of collinearity, consider (1) row-wise, i.e. $e_i^\top x_2 = \alpha x_1^\top e_i$ for some

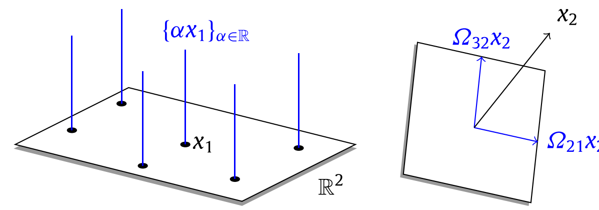

Fig. 2. On the left, the bundle (2) is depicted and on the right, two of the vectors $\Omega_{ij}x_2$, to which x_1 must be orthogonal to null the outputs (9), are plotted.

vector e_i from the standard basis of $\mathbb{R}^n$. But that α must be the same even if e_i is replaced by some e_j, i.e.

$$x_1^\top e_j e_i^\top x_2 = x_1^\top e_i e_j^\top x_2 \tag{4}$$

must hold for all i and j. The above equivalent characterizations of collinearity will all help us to characterize collinear dynamical systems in the course of the paper. Next, we precisely state what we mean by collinear dynamical systems.

Given a pair of coupled linear differential equations

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t), \quad x_1(0) \neq 0, \tag{5}$$
$$\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t), \quad x_2(0) \neq 0, \tag{6}$$

with all A_{ij} being $n \times n$ matrices, we ask whether collinearity of $x_1(0)$ and $x_2(0)$ is preserved under the flow of (5), (6). More particular, we say that the dynamical systems (5) and (6) are collinear if collinearity of their initial conditions causes their solutions to remain collinear for all times, i.e. if the implication

$$x_2(0) = \alpha(0)\,x_1(0) \;\Rightarrow\; \forall t \geq 0\, \exists \alpha(t):\; x_2(t) = \alpha(t)\,x_1(t)$$

holds true. Alternatively, reconsidering the fiber bundle (2), under which circumstances does the fact that $(x_1(0), x_2(0))$ lies in that bundle imply that $(x_1(t), x_2(t))$ remains in that bundle for all t? In terms of the equivalence relation (3), when does equivalence of initial conditions cause equivalence of solutions for all times, i.e., when is

$$x_1(0) \sim x_2(0) \quad \Rightarrow \quad \forall t \geq 0, \quad x_1(t) \sim x_2(t) \tag{7}$$

met? In other words, under which conditions is it possible to have (5), (6) defining a flow on $\mathbb{RP}^{n-1}$? For, if (5) and (6) are collinear, then their solutions define the same curve

$$t \mapsto [x_1(t)] = [x_2(t)] \tag{8}$$

on $\mathbb{RP}^{n-1}$, wherein $[\cdot]$ denotes an equivalence class of $\sim$. Last, still equivalent to the aforementioned definitions of collinear dynamical systems, making use of (4), when do the outputs

$$y_{ij}(t) = x_1(t) \cdot \Omega_{ij}x_2(t), \quad \Omega_{ij} = e_i e_j^\top - e_j e_i^\top, \quad j > i, \tag{9}$$

remain zero if they are initially zero (for a detailed account on such quadratic outputs, cf. [7] or [8])? Thereby, it shall be emphasized that the matrices $\Omega_{ij}, j > i$, generate the skew-symmetric matrices, i.e. the Lie algebra $\mathfrak{so}(n)$ of the special orthogonal group, whence we have that indeed any skew-symmetric bilinear form in $(x_1(t), x_2(t))$ remains zero for $t \geq 0$ if it was zero at $t = 0$. Two of the vectors $\Omega_{ij}x_2$, whose inner products with x_1 constitute y_{ij}, are depicted right in Fig. 2.

Having multiple equivalent definitions of collinear dynamical systems at hand, we are now ready to characterize collinear dynamical systems in the following section.