Systems & Control Letters 105 (2017) 78–83

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Tight bound for deciding convergence of consensus systems[☆]

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ARTICLE INFO

Article history:
Received 19 July 2016
Received in revised form 12 March 2017
Accepted 8 May 2017

Keywords:
Stability of matrix sets
Stochastic matrices
Consensus

ABSTRACT

We analyze the asymptotic convergence of all infinite products of matrices taken in a given finite set by looking only at *finite* or *periodic* products. It is known that when the matrices of the set have a common nonincreasing polyhedral norm, all infinite products converge to zero if and only if all infinite *periodic* products with period smaller than a certain value converge to zero. Moreover, bounds on that value are available (Lagarias and Wang, 1995).

We provide a stronger bound that holds for both polyhedral norms and polyhedral seminorms. In the latter case, the matrix products do not necessarily converge to zero, but all trajectories of the associated system converge to a common invariant subspace. We prove that our bound is tight for all seminorms.

Our work is motivated by problems in *consensus systems*, where the matrices are *stochastic* (nonnegative with rows summing to one), and hence always share a same common nonincreasing polyhedral seminorm. In that case, we also improve existing results.

1. Introduction

We consider the problem of determining the stability of matrix sets, that is, determining whether or not all infinite products of matrices from a given set converge to zero, or more generally to a common invariant subspace. This problem appears in several different situations in control engineering, computer science, and applied mathematics. For instance, the stability of matrix sets characterizes the stability of switching dynamical systems [1], which have numerous application in control [1–3]. Stability of matrix sets is instrumental in proving the continuity of certain wavelet functions [1,4]. Somewhat surprisingly, it also helped establishing the best known asymptotic bounds on the number of α-power-free binary words of length n, a central problem in combinatorics on words [5,6].

Deciding the stability of a matrix set is notoriously difficult and the decidability of this problem is not known. The related problem of the existence of an infinite product whose norm diverges is undecidable [7]. However, it is possible to decide stability when the set has the *finiteness property*, that is, when there is a bound p such that the existence of an infinite nonconverging product[1] implies the existence of an infinite nonconverging *periodic* product with period smaller than or equal to p. Indeed, checking the stability of the set can be done by checking the stability of all products whose length is smaller than or equal to p. In this work, we look for the smallest valid bound p.

A similar question is particularly relevant in the context of consensus problems. These systems are models for groups of agents trying to agree on some common value by an iterative process. Each agent has a value x_i which it updates by computing the weighted average of values of agents with which it can communicate. Consensus systems have attracted considerable attention due to their applications in control of vehicle formations [8], flocking [9,10] or distributed sensing [11,12]. They typically have time-varying communication networks due to e.g. communication failures, or to the movements of the agents. This leads to systems whose (linear) dynamics may switch at each time-step. When a set of possible linear dynamics is known, one fundamental question is whether the system converges for any switching sequence [13].

Consensus systems can be modeled by discrete-time linear switching systems, $x(t+1) = A_t x(t)$, where the transition matrices A_t are *stochastic* (nonnegative matrices whose rows sum to 1) because the agents always compute weighted averages. In this case, the products certainly do not converge to zero, since products of stochastic matrices remain stochastic. The central question is whether the agents asymptotically converge to the same value. Deciding whether a consensus system converges for any sequence of transition matrices and any initial condition corresponds to determining whether all left-infinite products of matrices taken from a set converge to a rank one matrix. Indeed, a stochastic

☆ R.M. Jungers is an F.R.S.-FNRS research associate. This work is also supported by the Belgian Network DYSCO, funded by the Belgian government and the Concerted Research Action (ARC) (13/18-054) of the French Community of Belgium.

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[1] I.e., an infinite product that does not converge to zero.

http://dx.doi.org/10.1016/j.sysconle.2017.05.001
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matrix is rank one if and only if all its rows are the same, and this situation corresponds to consensus. This particularization to stochastic matrices has other applications, including inhomogeneous Markov chains, and probabilistic automata [14].

Stochastic matrices share a *nonincreasing polyhedral seminorm* and this property provides important information on the asymptotic convergence of products of these matrices. Indeed, for sets of matrices sharing a common nonincreasing polyhedral seminorm, a bound p as discussed above is available. This was first established by Lagarias and Wang [15]. The authors also give an explicit value for p (namely half the number of faces of the unit ball of the norm). This result can easily be extended from norms to seminorms and we do so in the proof of Theorem 1.

The case of stochastic matrices has been analyzed earlier in the context of inhomogeneous Markov chains [4,14,16,17] and later in the context of consensus systems [13]. Paz [14] proved that all left-infinite products converge to a rank one matrix if and only if a certain condition on all products of length $B = \frac{1}{2}(3^n - 2^{n+1} + 1)$ is satisfied. In our recent paper [18], we showed that this bound can be derived from a generalization of the result of Lagarias and Wang applied to a particular seminorm.

Our contribution

In this article, we consider a general problem that includes these particular cases: we study matrix sets for which there exists a polyhedral seminorm which is nonincreasing for all matrices in the given set, and we wonder whether long products of these matrices are asymptotically contractive. We improve all the bounds previously known in the particular cases, and prove that our bound is tight. Inspired by ideas of Lagarias and Wang [15], we use an encapsulation of the convergence of the dynamical system in a *discrete* representation by a dynamical system on the face lattice of the polyhedral (semi)norm. Our results then rely on a careful study of the combinatorial structure of the trajectories in this discrete structure.

The improvement over the previously known bound depends on the seminorm. In the case of stochastic matrices, the improvement is a multiplicative factor of about $\frac{3}{2\sqrt{\pi n}}$.

2. Problem setting

Let $\Sigma = \{A_1, \ldots, A_m\}$ be a set of matrices and σ an infinite sequence of indices. We say that the product $\ldots A_{\sigma(2)}A_{\sigma(1)}$ is *periodic* if the sequence σ is periodic. We recall that a *seminorm* on $\mathbb{R}^n$ is a function $\|.\|$ with the following properties:

- $\forall x \in \mathbb{R}^n,\ a \in \mathbb{R},\ \|ax\| = |a|\|x\|$
- $\forall x, y \in \mathbb{R}^n,\ \|x + y\| \le \|x\| + \|y\|$.

We call a *polyhedral seminorm* a seminorm whose unit ball is a *polyhedron*, that is, a set that can be defined by a finite set of linear inequalities

$$\{x : \|x\| \le 1\} = \{x : \forall i,\ b_i^\top x \le c_i\}.$$

We say that a seminorm $\|.\|$ is *nonincreasing* with respect to a matrix A if

$$\forall x \in \mathbb{R}^n,\ \|Ax\| \le \|x\|.$$

Geometrically, this corresponds to its unit ball being invariant

$$A\{x : \|x\| \le 1\} \subseteq \{x : \|x\| \le 1\}.$$

We say that a seminorm is nonincreasing with respect to a set Σ of matrices if it is nonincreasing with respect to each of the matrices in Σ. We say that a matrix A *contracts* a seminorm $\|.\|$ if $\forall x \in \mathbb{R}^n$ s.t. $x \ne 0,\ \|Ax\| < \|x\|$. We say that an infinite product $\ldots A_{\sigma(2)}A_{\sigma(1)}$ contracts a seminorm $\|.\|$ if there is a t such that

$$A_{\sigma(t)} \ldots A_{\sigma(2)}A_{\sigma(1)}\{x : \|x\| \le 1\} \subset \operatorname{int}(\{x : \|x\| \le 1\}).$$

One can easily verify that if there is a p such that all products of length p of matrices in Σ contract a seminorm $\|.\|$, then all trajectories $x(t)$ of the corresponding switching system $x(t+1) = A_{\sigma(t)}x(t)$ asymptotically approach the set $\{x : \|x\| = 0\}$, and that their distance to that set decays exponentially as t increases. In particular, if $\|.\|$ is a norm, x converges exponentially to 0. In addition, if $\|.\|$ is the seminorm $\|x\|_{\mathcal{P}} = \frac{1}{2}(\max_i x_i - \min_i x_i)$ – a seminorm that is nonincreasing for stochastic matrices – then x approaches the *consensus space* $\{\alpha\mathbf{1}\}$. It is well known (see for example [18]) that each trajectory actually converges in that case to a specific (but possibly different) point in that set, as opposed to just approaching the set. For these reasons, we will investigate contraction of seminorms, keeping in mind that this question is intimately related to that of convergence.

Question 1. *Let $\|.\|$ be a polyhedral seminorm in $\mathbb{R}^n$ for some fixed n; what is the smallest p such that for any set Σ for which $\|.\|$ is nonincreasing, the existence of an infinite noncontracting product implies the existence of an infinite* periodic *noncontracting product with period smaller than or equal to p?*

3. The general case

In this section, we answer Question 1. We start by recalling some definitions (see [19] for more details). A *partially ordered set* or *poset* is a set P with a binary relation $\preceq$ that is transitive, antisymmetric and reflexive. We also note $x \prec y$ for the relation $x \preceq y$ and $x \ne y$. A poset $(P, \preceq)$ is called *graded* if it can be equipped with a rank function $r : P \mapsto \mathbb{N}$ such that $x \preceq y \Rightarrow r(x) \le r(y)$ and $(y \prec x$ and $\nexists z,\ y \prec z \prec x) \Rightarrow r(x) = r(y) + 1$. The set of all elements of a given rank is called a rank level. A poset is called a *lattice* if any pair of elements has a unique infimum and a unique supremum.

Intuitively, a *face* is the generalization of a vertex (or an edge, or a facet) to an arbitrary dimension. The formal definition is the following.

Definition 1 (*Faces of a Polyhedron*). A nonempty subset F of an n-dimensional polyhedron $\mathcal{Q}$ is called a *face* or *closed face* if one of the following holds:

- $F = \mathcal{Q}$,
- $F = \varnothing$
- or F can be represented as $F = \mathcal{Q} \cap \{x : b^\top x = c\}$ where $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ are such that

  $\forall x \in \mathcal{Q},\ b^\top x \le c.$

If the face contains exactly $d + 1$ affinely independent points,[2] we call d the *dimension* of the face. A *proper face* is a face that is neither the polyhedron itself nor the empty face. An *open face* is the relative interior of a face. Finally, a *facet* is a face of dimension $n - 1$.

It is well known that faces of any dimension are intersections of facets and their number is therefore finite. It is also known that any polyhedron decomposes into a disjoint union of open faces.

We use the term *double-face* to denote the set $F \cup -F$, for some proper face F. A double-face is called open if the face F is open, and closed otherwise.

[2] The points $u_0, u_1, \ldots, u_d$ are called affinely independent if $u_1 - u_0, u_2 - u_0, \ldots$ are linearly independent.