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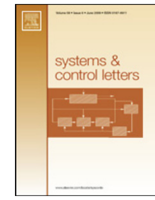

A converse to the deterministic separation principle

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ABSTRACT

In the classical theory of finite-dimensional linear time-invariant systems in state space form the term *deterministic separation principle* refers to the observation that a stabilizing output feedback controller can be constructed by first constructing an asymptotic state observer that is then coupled to a stabilizing state feedback controller. In this paper we discuss the following *converse* problem: Can *every* stabilizing output feedback controller be realized as interconnection of an asymptotic state observer and a stabilizing state feedback controller? We will provide an affirmative answer to this question (modulo a number of technicalities) in a behavioral setting and with the help of rational representations.

1. Introduction

The classical deterministic separation principle says that, given the plant

$$\dot{x} = Ax + Bu,$$
$$y = Cx,$$

an asymptotic full state observer

$$\dot{\hat{x}} = (A - GC)\hat{x} + Bu + Gy \tag{1}$$

with $A - GC$ Hurwitz, and a stabilizing static full state feedback controller

$$u = F\hat{x} \tag{2}$$

with $A + BF$ Hurwitz, the closed-loop dynamics is given by

$$\begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} A + BF & BF \\ 0 & A - GC \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix}, \tag{3}$$

where $e = \hat{x} - x$ is the observer error [1]. It follows from the form of the system matrix in (3) that $\lim_{t\to\infty} x(t) = 0$, i.e. the observer-based output feedback controller (1) and (2) is stabilizing. In fact, it is even *internally* or *totally* stabilizing since $x \to 0$ implies both $y \to 0$ and $\hat{x} \to 0$ (since also $e \to 0$), and hence also $u \to 0$.

The above principle is called a *separation* principle because it allows to complete the task of constructing an output feedback controller with desirable properties (namely stability) by *separately* constructing a state observer and a full state feedback controller with that property. Another classical but unrelated separation principle is that of optimal stochastic control, see e.g. [2].

An obvious converse question is whether there are any *other* constructions of (totally) stabilizing output feedback controllers, or whether *any* such controller permits an interpretation as a series connection of a full state observer followed by a (possibly dynamic) full state feedback controller. A partial answer to this question was given by Schumacher using the geometric notion of compensator couples at the beginning of the 1980s [3], but a full answer remained elusive to this date.

In this paper we address the converse question in a behavioral framework using both polynomial and rational representations of linear differential systems. We show that, under mild assumptions on the to be controlled system with variables (x, u, y), any controllable, regular, totally stabilizing controller through the variables (u, y) can be separated into an asymptotic i/o-observer for x from (u, y) with variables $(\hat{x}, u, y)$ and a regular, totally stabilizing controller with variables $(\hat{x}, u)$ in the sense that the controllable part of the observer/$(\hat{x}, u)$-controller interconnection coincides with the given (u, y)-controller.

The paper is organized as follows. Section 2 introduces our notation and collects relevant results from the theory of behaviors including basics on rational representations. In Section 3 we review the required material on stabilization in a behavioral framework. In Section 4, we develop a convenient system representation that

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is adapted to the problem treated in this paper. Section 5 contains the main result, Section 6 discusses the special case of state space systems and Section 7 concludes the paper.

2. Behaviors of linear differential systems

In this paper we will make heavy use of the mathematical machinery of the behavioral approach to linear differential systems. A linear differential system is defined as a triple $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ whose *behavior* $\mathfrak{B} \subset \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ is the solution space of a finite set of higher order constant coefficient linear differential equations.

2.1. Polynomial and rational kernel representations

Behaviors of linear differential systems can be represented in terms of a real polynomial matrix $R(s)$ with $\mathtt{w}$ columns as $R(\frac{d}{dt})w = 0$, so that

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{d}{dt}\right) w = 0 \right\}. \tag{4}$$

The representation (4) is called a *polynomial kernel representation* of $\mathfrak{B}$, and we often write $\mathfrak{B} = \ker R(\frac{d}{dt})$. If R_1 and R_2 are two full row rank polynomial matrices, then they represent the same behavior $\mathfrak{B}$, i.e. $\mathfrak{B} = \ker R_1(\frac{d}{dt}) = \ker R_2(\frac{d}{dt})$, if and only if there exists a unimodular polynomial matrix U such that $R_2 = UR_1$. For an extensive treatment of polynomial representations of behaviors we refer to [4].

Behaviors also admit representations in terms of real rational matrices. A detailed exposition on rational representations can be found in [5]. Here we will give a brief review. Recall that any given real rational matrix admits a left coprime factorization into polynomial matrices. A factorization of a real rational matrix R as $R = P^{-1}Q$ with P, Q real polynomial matrices is called a left coprime factorization if $\begin{pmatrix} P & Q \end{pmatrix}$ is left prime (meaning that it has a polynomial right inverse) and $\det(P) \neq 0$. Following [5], if $R = P^{-1}Q$ is such a left coprime factorization then we *define* w to be a solution of $R(\frac{d}{dt})w = 0$ if it is a solution of the differential equation $Q(\frac{d}{dt})w = 0$. In other words, we define

$$\ker R\left(\frac{d}{dt}\right) := \ker Q\left(\frac{d}{dt}\right), \tag{5}$$

which is well-defined since any two left coprime factorizations of R differ by a unimodular polynomial factor. For a given rational matrix R, we call a representation of $\mathfrak{B}$ as $R(\frac{d}{dt})w = 0$ a *rational kernel representation* of $\mathfrak{B}$ and write $\mathfrak{B} = \ker R(\frac{d}{dt})$. For additional material on rational representations we refer to [6,7]. In this paper we will often assume that the rational matrices $R(s)$ used in kernel representations have full row rank over the field of real rational functions. This is equivalent to saying that the kernel representation is minimal, see [5,6].

As noted before, two minimal polynomial kernel representations differ by a unimodular polynomial factor. A similar statement does *not* hold for rational representations. We will come back to this in the next subsection.

2.2. Controllability and controllable part

In the behavioral approach an important role is played by the property of *controllability*. The definition of controllability of a behavior $\mathfrak{B}$ is well known, and can be found in [4]. Controllability of a behavior can be tested in terms of its full row rank rational kernel representations as follows: If $\mathfrak{B} = \ker R(\frac{d}{dt})$ where $R(s)$ is a rational matrix, then $\mathfrak{B}$ is controllable if and only if R has no zeros.

A behavior $\mathfrak{B}$ is called *autonomous* if it is a finite dimensional subspace of $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. In terms of its rational kernel representations $\mathfrak{B} = \ker R(\frac{d}{dt})$ this property requires that R has full column rank.

Any behavior $\mathfrak{B}$ admits a direct sum decomposition as $\mathfrak{B} = \mathfrak{B}_{\text{cont}} \oplus \mathfrak{B}_{\text{aut}}$, where $\mathfrak{B}_{\text{cont}}$, called *the controllable part* of $\mathfrak{B}$, is the largest controllable subbehavior of $\mathfrak{B}$, and $\mathfrak{B}_{\text{aut}}$, called *an autonomous part*, is an autonomous subbehavior of $\mathfrak{B}$. The controllable part is uniquely determined by $\mathfrak{B}$. In terms of its rational kernel representations $R(\frac{d}{dt})w = 0$, the controllable part of $\mathfrak{B}$ can be found by factorizing $R = Q\bar{R}$ with Q nonsingular rational and $\bar{R}$ a left prime polynomial matrix. For any such factorization we have $\mathfrak{B}_{\text{cont}} = \ker \bar{R}(\frac{d}{dt})$, see [5].

It was shown in [6] that if R_1 and R_2 are full row rank rational matrices, then there exists a nonsingular rational matrix Q such that $R_2 = QR_1$ if and only if R_1 and R_2 represent behaviors with the same controllable part, i.e. $(\ker R_1(\frac{d}{dt}))_{\text{cont}} = (\ker R_2(\frac{d}{dt}))_{\text{cont}}$.

2.3. Elimination of variables

We will now review the basics of elimination of variables. Suppose we have a behavior $\mathfrak{B}$ in which the manifest variable is partitioned into two parts as $w = (v, c)$. Let $R_v(\frac{d}{dt})v + R_c(\frac{d}{dt})c = 0$ be a polynomial kernel representation of $\mathfrak{B}$. The space of trajectories that satisfies this equation is called the *full behavior*. The space of tractories c that are compatible with the equation of the full behavior is called *the behavior with v eliminated* and is given by

$$\mathfrak{B}_c := \left\{ c \mid \text{ there exists } v \text{ such that } R_v\left(\frac{d}{dt}\right) v + R_c\left(\frac{d}{dt}\right) c = 0 \right\}. \tag{6}$$

The elimination problem is to obtain a kernel representation of (6). Such a kernel representation can be obtained as follows: first find a unimodular polynomial matrix U such that

$$UR_v = \begin{pmatrix} R_{v,1} \\ 0 \end{pmatrix}$$

where $R_{v,1}$ has full row rank. Next, apply the same unimodular matrix to R_c to obtain

$$UR_c = \begin{pmatrix} R_{c,1} \\ R_{c,2} \end{pmatrix}.$$

Since $\ker \begin{pmatrix} R_v & R_c \end{pmatrix} (\frac{d}{dt}) = \ker U \begin{pmatrix} R_v & R_c \end{pmatrix} (\frac{d}{dt})$, a new, more structured, polynomial kernel representation of $\mathfrak{B}$ is then given by

$$\begin{pmatrix} R_{v,1} & R_{c,1} \\ 0 & R_{c,2} \end{pmatrix} \begin{pmatrix} v \\ c \end{pmatrix} = 0.$$

A kernel representation of the behavior (6) is now given by $R_{c,2}(\frac{d}{dt})c = 0$ (see [4]).

The above construction to obtain the eliminated behavior (6) is only valid for polynomial kernel representations and uses unimodular premultiplication. Its counterpart for the case that we deal with rational kernel representations and, instead of unimodular premultiplication, we use premultiplication with a nonsingular rational matrix is more subtle and is dealt with in the following lemma.

Lemma 2.1. *Let $R_v(\frac{d}{dt})v + R_c(\frac{d}{dt})c = 0$ be a rational kernel representation of $\mathfrak{B}$. Let Q be a nonsingular rational matrix such that*

$$QR_v = \begin{pmatrix} R_{v,1} \\ 0 \end{pmatrix}$$