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Modeling of physical network systems

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ABSTRACT

Conservation laws and balance equations for physical network systems typically can be described with the aid of the incidence matrix of a directed graph, and an associated symmetric Laplacian matrix. Some basic examples are discussed, and the extension to k-complexes is indicated. Physical distribution networks often involve a non-symmetric Laplacian matrix. It is shown how, in case the connected components of the graph are strongly connected, such systems can be converted into a form with balanced Laplacian matrix by constructive use of Kirchhoff's Matrix Tree theorem, giving rise to a port-Hamiltonian description. Application to the dual case of asymmetric consensus algorithms is given. Finally it is shown how the minimal storage function for physical network systems with controlled flows can be explicitly computed.

1. Introduction

The topic of physical network systems has been always dear to Jan Willems' heart, from his early work on network synthesis and physical systems theory to his seminal work on dissipativity theory [1,2], and from the initial developments in behavioral theory to more recent 'educational' papers [3,4]. I was often fortunate to witness these scientific developments from a close distance, and to be involved in penetrating discussions with Jan. Many of these animated debates centered around the 'right' and 'ultimate' definition of the basic concepts. Needless to say that my own ideas, including the ones presented in this paper, have been heavily influenced by Jan's.

The structure of the paper is as follows. In Section 3, after a recap of basic notions in algebraic graph theory in Section 2, I will discuss how *conservation laws* and *balance equations* for physical network are often naturally expressed in terms of the incidence matrix of a directed graph, and how this leads to a well-defined class of systems involving a *symmetric Laplacian matrix*. Next, in Section 4, attention will be directed to a more general class of physical network systems, of general distribution type, where the Laplacian matrix is *not* necessarily symmetric. Under the assumption of strong connectedness it will be shown how by means of Kirchhoff's Matrix Tree theorem the system can be constructively converted into a system with *balanced* Laplacian matrix, admitting a stability analysis similar to the symmetric case. Section 5 is devoted to the analysis of *available storage* of passive physical network systems; a fundamental concept introduced in Jan Willems' seminal paper [1]. Section 6 contains conclusions.

2. Preliminaries about graphs

We recall from e.g. [5,6] a few standard definitions and facts. A *graph* $\mathcal{G}(\mathcal{V},\mathcal{E})$, is defined by a set $\mathcal{V}$ of *vertices* (nodes) and a set $\mathcal{E}$ of *edges* (links, branches), where $\mathcal{E}$ is identified with a set of unordered pairs $\{i,j\}$ of vertices $i,j\in\mathcal{V}$. We allow for multiple edges between vertices, but not for self-loops $\{i,i\}$. By endowing the edges with an orientation we obtain a *directed graph*. A directed graph with n vertices and m edges is specified by its $n\times m$ *incidence matrix*, denoted by D. Every column of D corresponds to an edge of the graph, and contains exactly one -1 at the row corresponding to its tail vertex and one $+1$ at the row corresponding to its head vertex, while the other elements are 0. In particular, $\mathbb{1}^T D = 0$ where $\mathbb{1}$ is the vector of all ones. Furthermore, $\ker D^T = \operatorname{span}\mathbb{1}$ if and only if the graph is *connected* (any vertex can be reached from any other vertex by a sequence of, - undirected -, edges). In general, the dimension of $\ker D^T$ is equal to the number of connected components. A graph is *strongly connected* if any vertex can be reached from any other vertex by a sequence of directed edges. For any diagonal positive semi-definite $m\times m$ matrix R we define a *symmetric Laplacian matrix* of the graph as $L := DRD^T$, where the positive diagonal elements $r_1,\dots,r_m$ of the matrix R are the weights of the edges. It is well-known [5] that L is *independent* of the orientation of the graph.

The vertex space [7] Λ_0 is defined as the set of all functions from the vertex set $\mathcal{V}$ to $\mathbb{R}$. Obviously Λ_0 can be identified with $\mathbb{R}^n$. The dual space of Λ_0 is denoted by Λ^0. Furthermore, the edge space Λ_1

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is defined as the linear space of functions from the edge set $\mathcal{E}$ to $\mathbb{R}$, with dual space denoted by Λ^1. Both spaces can be identified with $\mathbb{R}^k$. It follows that the incidence matrix D defines a linear map (denoted by the same symbol) $D : \Lambda_1 \to \Lambda_0$ with adjoint map $D^T : \Lambda^0 \to \Lambda^1$. Using these abstractions it is straightforward to extend the physical network dynamics described in this paper to other spatial domains than $\mathbb{R}$. Indeed, for any linear space $\mathcal{R}$ (e.g., $\mathcal{R} = \mathbb{R}^3$) we can define Λ_0 as the set of functions from $\mathcal{V}$ to $\mathcal{R}$, and Λ_1 as the set of functions from $\mathcal{E}$ to $\mathcal{R}$. In this case we can identify Λ_0 with the tensor product $\mathbb{R}^n \otimes \mathcal{R}$ and Λ_1 with the tensor product $\mathbb{R}^k \otimes \mathcal{R}$. Furthermore, the incidence matrix D defines a linear map $D \otimes I : \Lambda_1 \to \Lambda_0$, where I is the identity map on $\mathcal{R}$. In matrix notation $D \otimes I$ equals the Kronecker product of the incidence matrix D and the identity matrix I. See [7] for further details.

3. Physical network systems with symmetric Laplacian matrices

The structure of physical network dynamics is usually based on *conservation laws* and *balance equations*. Given a directed graph $\mathcal{G}$ with incidence matrix D the basic way of expressing conservation laws is by equations of the form

$$Df + f_S = 0, \tag{1}$$

where $f \in \Lambda_1 \simeq \mathbb{R}^m$ is the vector of *flows* through the edges of the graph, and $f_S \in \Lambda_0 \simeq \mathbb{R}^n$ is the vector of *injected flows* at the vertices. This can be regarded as a generalized form of Kirchhoff's current laws, in which case f denotes the vector of currents through the edges of the electrical circuit graph, and f_S are additional currents injected at the vertices of the circuit graph.[1] Its restricted form is $Df = 0$ (no flows/currents injected at the vertices).

The injected flows at the vertices either correspond to external flows or to *storage* at the vertices, in which latter case there are state variables $x_i \in \mathbb{R}$ (or, see above, x_i belonging to a general linear space $\mathcal{R}$) associated to each ith vertex, corresponding to $\dot{x} = -f_S$. This leads to the differential equations

$$\dot{x} = Df, \tag{2}$$

expressing the basic *conservation laws* of the system: the sum of the incoming and outgoing flows through the edges incident to the ith vertex is equal to the rate of storage at that vertex.

Often, the flows $f \in \Lambda_1$ through the edges are determined by *efforts* $e \in \Lambda^1$ associated to the edges, through a *resistive relation* of the form $f = -R(e)$, for some map $R : \Lambda^1 \to \Lambda_1$ satisfying $e^T R(e) \geq 0$, which is usually *diagonal* in the sense that its jth component only depends on the effort e_j associated to the jth edge. The components of e thus can be regarded as 'driving forces' for the flows f. In the linear case $f = -Re$ with R a diagonal $n \times n$ matrix with nonnegative diagonal elements (in an electrical circuit context corresponding to conductances of resistors at the edges).

In many cases of interest, the effort variable e_j corresponding to the jth edge is an *across* variable which is determined by the *difference* of effort variables at the vertices incident to that edges, i.e.,

$$e = D^T e_S, \tag{3}$$

with $e_S \in \Lambda^0$ the vector of effort variables at the vertices. This corresponds to a *balance law* or an *equilibrium condition*: the driving force e_j for the flow through the jth edge is zero whenever the efforts at the vertices incident to this edge are equal. (In an electrical circuit e_S corresponds to the voltage potentials at the vertices, and e to the voltages across the edges.)

Typically[2] the efforts e_S at the vertices are determined by the state variables x following

$$e_S = \frac{\partial H}{\partial x}(x), \tag{4}$$

where $H : \Lambda_0 \to \mathbb{R}$ is the *total stored energy* at the vertices. Usually, H is an *additive* energy function $H(x) = H_1(x_1) + \cdots + H_n(x_n)$. This leads to the equation

$$\dot{x} = -DRD^T \frac{\partial H}{\partial x}(x). \tag{5}$$

The $n \times n$ matrix $L := DRD^T$ is a *symmetric Laplacian* matrix, that is a symmetric matrix with nonnegative diagonal elements and nonpositive off-diagonal elements whose column and row sums are zero. Conversely, any symmetric Laplacian matrix can be represented as $L = DRD^T$ for some incidence matrix D and positive diagonal matrix R. Clearly L is positive semi-definite.

Eq. (5) is the common form of physical network systems with energy storage confined to the vertices. (See [7] for other cases, in particular including energy storage associated to the edges.) They can be immediately seen to be in *port-Hamiltonian* form. Recall, see e.g. [9,10], that port-Hamiltonian systems with inputs and outputs, in the absence of algebraic constraints and having linear energy-dissipating relations, are given by equations of the form

$$\begin{aligned} \dot{x} &= [\mathcal{J}(x) - \mathcal{R}(x)] \frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x) \frac{\partial H}{\partial x}(x), \end{aligned} \tag{6}$$

with $\mathcal{J}(x) = -\mathcal{J}^T(x)$ and $\mathcal{R}(x) = \mathcal{R}^T(x) \geq 0$. The system (5) is obviously port-Hamiltonian (without inputs and outputs) with $\mathcal{J}(x) = 0$ and $\mathcal{R}(x) = L = DRD^T$.

Example 3.1 (*Mass–damper Systems*)**.** A paradigmatic example of the above scenario is a linear *mass–damper system*

$$\dot{p} = -DRD^T M^{-1} p, \tag{7}$$

with p the vector of *momenta* of the masses associated to the vertices, M the diagonal mass matrix, R the diagonal matrix of damping coefficients of the dampers attached to the edges, and $H(p) = \frac{1}{2} p^T M^{-1} p$ the total kinetic energy of the masses. The vector of velocities $v = M^{-1} p$ converges to a vector in the kernel of $L = DRD^T$. In particular, if the graph is weakly connected the vector v converges to a vector of the form $v^* \mathbb{1}$, with $v^* \in \mathbb{R}$ (equal velocities). For extensions to mass–spring–damper systems see [7].

Example 3.2 (*Hydraulic Networks*)**.** Consider a hydraulic network between n fluid reservoirs whose storage is described by the elements of a vector x. Mass balance corresponds to $\dot{x} = Df$ where $f \in \mathbb{R}^k$ is the flow through the k pipes linking the reservoirs. Let each storage variable x_i determine a pressure $\frac{\partial H_i}{\partial x_i}(x_i)$ for a certain energy function H_i. Assuming that the flow f_j is proportional to the difference between the pressure of the head reservoir and the pressure of the tail reservoir this leads to Eq. (5).

Example 3.3 (*Symmetric Consensus Algorithms*)**.** Eq. (5) for $H(x) = \frac{1}{2}\|x\|^2$ reduces to $\dot{x} = -Lx, L = DRD^T$, which is the standard symmetric consensus protocol in continuous time, with weights given by the diagonal elements of R. In Section 4.3 we will pay attention to *asymmetric* consensus dynamics.

[1] Indeed, the presence of flows injected at the vertices is essential in Kirchhoff's original paper [8].

[2] The electrical circuit case is somewhat different, since resistors, inductors, and capacitors are all associated with the edges, and thus there is no storage at the vertices. Storage of charge at the vertices would correspond to *grounded capacitors*, with the ground node not included in the set of vertices.

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