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# Data-driven control: A behavioral approach

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ABSTRACT

In this work, we study the design of a controller using system data. We present three data-driven approaches based on the notion of control as interconnection. In the first approach, we use both the data and representations to compute control variable trajectories that impose a prescribed path on the to-be-controlled variables. The second method is completely data-driven and we prove sufficient conditions for determining a controller directly from data. Finally, we show how to determine a controller directly from data in the case where the control and to-be-controlled variables coincide.



## 1. Introduction

Over the years, several authors have proposed different methods for using system data in the design of a controller. For example, in [1–3] system data is used to find suitable control *inputs* and in [4] data is used to falsify a control law. Furthermore, data-driven control techniques have been applied in different applications and processes such as real-time, fault-tolerant controller design for electrical circuits [5], on-line data-driven control switching [4] and data-driven fault tolerant control design, see [6].

In this paper, we show how to find a controller directly using system data. Our solutions are based on the *behavioral framework* like in [3], but we do not assume a priori an input/output partition of variables. We use the *interconnection paradigm*, see [7,8]. Most importantly, in our approach one can also identify a controller *representation* under suitable conditions which will be specified, while in [3] the aim was to design a control *input*. Furthermore, we do not have a prior assumption that the set of admissible control laws is known, as in [4]. Our solutions are off-line, non-iterative and summarized by a step-by-step algorithm.

This paper is organized as follows. In Section 2, we cover some relevant background material. In Section 3, we state formally the problems solved in this paper. In Sections 4–6, we present our solutions. In Section 7, we provide some conclusions. All the necessary lemmas and proofs are gathered in Appendices A and B, respectively.

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**Notation.** $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the set of real numbers, complex numbers, integers and positive integers, respectively. The space of $\mathtt{w}$ dimensional real vectors is denoted by $\mathbb{R}^{\mathtt{w}}$ and that of $\mathtt{g} \times \mathtt{w}$ real matrices by $\mathbb{R}^{\mathtt{g}\times\mathtt{w}}$. When both dimensions are not specified but finite, we write $\mathbb{R}^{\bullet\times\bullet}$. The space of real matrices with $\mathtt{g}$ rows and an infinite number of columns is denoted by $\mathbb{R}^{\mathtt{g}\times\infty}$. $I_{\mathtt{w}}$, $0_{\mathtt{w}\times\mathtt{w}}$ denotes $\mathtt{w}\times\mathtt{w}$ identity and zero matrices, respectively. colspan$(A)$ and leftkernel$(A)$ denotes the column span of $A \in \mathbb{R}^{\bullet\times\bullet}$ and the subspace spanned by all vectors $v$ such that $vA = 0$, respectively. col$(A, B)$ is the matrix obtained by stacking $A \in \mathbb{R}^{\bullet\times\mathtt{w}}$ over $B \in \mathbb{R}^{\bullet\times\mathtt{w}}$, and $\mathrm{col}(A_i)_{i=1,\ldots,l} := \mathrm{col}(A_1, \ldots, A_l)$. The ring of polynomials with real coefficients in the indeterminate $\xi$ is denoted by $\mathbb{R}[\xi]$ and the set of $\mathtt{g}\times\mathtt{w}$ matrices in the indeterminate $\xi$ is denoted by $\mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$. Let $R = R_0 + \cdots + R_L\xi^L \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}$ with $R_L \neq 0$ then $L$ is the degree of $R$ and is denoted by deg$(R)$. $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$, is closely associated with the *coefficient matrix* $\tilde{R} := [R_0 \ \ldots \ R_L \ 0_{\mathtt{g}\times\mathtt{w}} \ \ldots\ldots]$. $\tilde{R}$ has an infinite number of columns, which are zero everywhere except for a finite number of elements. Notice that $R = \tilde{R}\mathrm{col}(I_{\mathtt{w}} \ \ldots \ I_{\mathtt{w}}\xi^L \ 0 \ \ldots)$. $\sigma_R\tilde{R} := [0_{\mathtt{g}\times\mathtt{w}} \ R_0 \ \ldots \ R_L \ 0_{\mathtt{g}\times\mathtt{w}} \ \ldots]$ is the *right shift* of $\tilde{R}$ and $\sigma_R^k\tilde{R}$ denotes $k$ right shifts of $\tilde{R}$ where $k \in \mathbb{Z}_+$. The set of all maps from $\mathbb{Z}$ to $\mathbb{R}$ is denoted by $(\mathbb{R})^{\mathbb{Z}}$. The collection of all linear, closed, shift invariant subspaces of $(\mathbb{R}^{\bullet})^{\mathbb{Z}}$ equipped with the topology of pointwise convergence is denoted by $\mathscr{L}^{\bullet}$. The *backward shift* operator $\sigma$ is defined by $(\sigma f)(t) := f(t+1)$.

## 2. Linear discrete complete system

We define a *dynamical system* by $\Sigma := (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ with $\mathbb{Z}$ the *time axis*, $\mathbb{R}^{\mathtt{w}}$ the *signal space* and $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ the *behavior*. Let

$\Delta \in \mathbb{Z}_+$, then the restriction of $\mathfrak{B}$ on the interval $[1, \Delta]$ is defined by

$$\mathfrak{B}_{|[1,\Delta]} := \{w : [1, \Delta] \to \mathbb{R}^{\mathtt{w}} | \exists\, w' \in \mathfrak{B} \text{ s.t. } w(t) = w'(t) \text{ for all } 1 \leqslant t \leqslant \Delta\}.$$

$\Sigma$ is *linear* if $\mathfrak{B}$ is a linear subspace of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$, *time-invariant* if $\sigma\mathfrak{B} \subseteq \mathfrak{B}$ and *complete* if $[w \in \mathfrak{B}] \Leftrightarrow [w_{|[1,\Delta]} \in \mathfrak{B}_{|[1,\Delta]}$ for all $\Delta \in \mathbb{Z}]$. Moreover, $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}$ if and only if there exists $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ such that $\mathfrak{B} := \{w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} | R(\sigma)w = 0\}$, i.e. $\mathfrak{B} = \ker(R(\sigma))$. $R$ is called a *kernel representation* of $\mathfrak{B}$ and is *minimal* if no other kernel representation of $\mathfrak{B}$ has less than g rows. $\Sigma_{\mathrm{L}} := (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{1}, \mathfrak{B}_{full})$ is a dynamical system with *latent variables*. $\mathfrak{B}_{full}$ is called the *full behavior* and consists of all trajectories $(w, \ell)$ with $w$ a *manifest variable* trajectory and $\ell$ a *latent variable* trajectory. Let $R \in \mathbb{R}^{\mathtt{d}\times\mathtt{w}}[\xi]$ and $M \in \mathbb{R}^{\mathtt{d}\times 1}[\xi]$ then $\mathfrak{B}_{full} \in \mathscr{L}^{\mathtt{w}+1}$ admits a representation of the form $R(\sigma)w = M(\sigma)\ell$, called a *hybrid representation*. It has been shown in [9] that $\mathfrak{B}_{full}$ induces a *manifest behavior* defined by $\mathfrak{B} := \{w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}} \mid \exists \ell \in (\mathbb{R}^{1})^{\mathbb{Z}} \text{ s.t. } (w, \ell) \in \mathfrak{B}_{full}\}$. $\mathfrak{B}$ is obtained by using the projection operator $\pi_w : (\mathbb{R}^{\mathtt{w}} \times \mathbb{R}^{1})^{\mathbb{Z}} \to (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ defined by $w := \pi_w(w, \ell)$, hence $\mathfrak{B} = \pi_w(\mathfrak{B}_{full})$.

Let $w_1, w_2 \in \mathfrak{B}$, then $\mathfrak{B}$ is *controllable* if there exists $t_1 \geq 0$ and $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t \leq 0$ and $w(t) = w_2(t - t_1)$ for $t \geq t_1$. Equivalently, $\mathfrak{B} = \ker(R(\sigma))$ is controllable if and only if $R(\lambda)$ is full row rank for all $\lambda \in \mathbb{C}$. We denote by $\mathscr{L}^{\mathtt{w}}_{contr}$ the collection of all controllable elements of $\mathscr{L}^{\mathtt{w}}$. Let $(w_1, w_2) \in \mathfrak{B}$, $w_2$ is *observable* from $w_1$ if there exists $f : (\mathbb{R}^{\mathtt{w}_1})^{\mathbb{Z}} \to (\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}}$ such that $w_2 = f(w_1)$. Let $\mathfrak{B}$ be described by $R_1(\sigma)w_1 = R_2(\sigma)w_2$, with $R_1 \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}_1}[\xi]$ and $R_2 \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}_2}[\xi]$, then $w_2$ is observable from $w_1$ if and only if $R_2(\lambda)$ is full column rank for all $\lambda \in \mathbb{C}$, see [10].

$\mathfrak{B}$ is associated with a number of integer invariants, [10]. The following are of interest in this paper. Let $w \in \mathfrak{B}$, then a partition of $w := (w_1, w_2)$ is an *input/output* partition if $w_1$ is *maximally free*, i.e. $\pi_{w_1}(\mathfrak{B}) = (\mathbb{R}^{\bullet})^{\mathbb{Z}}$ and $w_2$ contains no free components. $w_1$ is the input and $w_2$ output. We denote by $\mathtt{p}(\mathfrak{B})$ and $\mathtt{m}(\mathfrak{B})$ the *output* and *input cardinality* (the number of outputs or inputs), respectively. The smallest integer $L$ such that $[w_{|[t,t+L]} \in \mathfrak{B}_{|[t,t+L]}$ for all $t \in \mathbb{Z}] \Rightarrow [w \in \mathfrak{B}]$ is called the *lag* and denoted by $\mathrm{L}(\mathfrak{B})$. $\mathtt{n}(\mathfrak{B})$ denotes the *McMillan degree*, i.e. the smallest state-space dimension among all possible state representations of $\mathfrak{B}$. Finally, $\mathtt{l}(\mathfrak{B})$ denotes the *shortest lag* described as follows. Let $\mathfrak{B} = \ker(R(\sigma))$ and define the degree of each row of $R$ to be the largest degree of the entries. Then the minimum of degrees of the rows of $R$ is the minimal lag associated with $R$. $\mathtt{l}(\mathfrak{B})$ is smallest possible minimal lag over all $R$ such that $\mathfrak{B} = \ker(R(\sigma))$.

### 2.1. Annihilators and fundamental lemma

The module of *annihilators* associated with $\mathfrak{B}$ is defined by $\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{1\times\mathtt{w}}[\xi] | n(\sigma)\mathfrak{B} = 0\}$. If $\mathfrak{B} = \ker(R(\sigma))$ then $\mathfrak{N}_{\mathfrak{B}}$ equals the $\mathbb{R}[\xi]$-submodule of $\mathbb{R}^{1\times\mathtt{w}}[\xi]$ generated by the rows of $R$, see [11]. We denote the set of annihilators of $\mathfrak{B}$ of degree less than $j \in \mathbb{Z}_+$ by $\mathfrak{N}^{j}_{\mathfrak{B}} := \{r \in \mathbb{R}^{1\times\mathtt{w}}[\xi] | r \in \mathfrak{N}_{\mathfrak{B}}$ and $r$ has degree $\leqslant j\}$. Let $r_1, \ldots, r_i \in \mathfrak{N}^{j}_{\mathfrak{B}}$ and $\tilde{r}_1 \ldots \tilde{r}_i$ be the coefficients of $r_1, \ldots, r_i$; then $\tilde{\mathfrak{N}}^{j}_{\mathfrak{B}}$ denotes the set containing $\tilde{r}_1 \ldots \tilde{r}_i$.

**Definition 1.** Let $L \in \mathbb{Z}_+$. The *Hankel matrix* associated with a vectors $w(1), \ldots, w(T)$ for $T > L$ is defined by

$$\mathfrak{H}_L(w) := \begin{bmatrix} w(1) & w(2) & \cdots & w(T-L+1) \\ w(2) & w(3) & \cdots & w(T-L+2) \\ \vdots & \vdots & \cdots & \vdots \\ w(L) & w(L+1) & \cdots & w(T) \end{bmatrix}.$$

$\mathfrak{H}_{L,J}(w)$ is the *Hankel matrix* with $L$ block rows and $J$ columns.

**Definition 2.** A vector $\tilde{u} = \tilde{u}(1), \tilde{u}(2), \ldots, \tilde{u}(T)$ is *persistently exciting* of order $L$ if $\mathfrak{H}_L(\tilde{u})$ is full row rank.

Now we state the "*fundamental lemma*" cf. [12].

**Lemma 1.** *Assume $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}_{contr}$. Let $\tilde{w} = \tilde{w}(1), \tilde{w}(2), \ldots, \tilde{w}(T) := col(\tilde{u}, \tilde{y}) \in \mathfrak{B}_{|[1,T]}$ such that $\tilde{u}(k) \in \mathbb{R}^{\mathtt{m}(\mathfrak{B})}$ is an input and $\tilde{y}(k) \in \mathbb{R}^{\mathtt{p}(\mathfrak{B})}$ an output, for $1 \leqslant k \leqslant T$. Finally, let $L \in \mathbb{Z}_+$ be such that $L > \mathrm{L}(\mathfrak{B})$. If $\tilde{u}$ is persistently exciting of order at least $L + \mathtt{n}(\mathfrak{B})$, then colspan$(\mathfrak{H}_L(\tilde{w})) = \mathfrak{B}_{|[1,L]}$ and leftkernel$(\mathfrak{H}_L(\tilde{w})) = \tilde{\mathfrak{N}}^{L}_{\mathfrak{B}}$.*

**Proof.** See Theorem 1 of [12].

Under the conditions of Lemma 1, then for all $\tilde{w}' \in \mathfrak{B}_{|[1,L]}$ there exists $\tilde{\upsilon} \in \mathbb{R}^{T-L+1}$ such that $\tilde{w}' = \mathfrak{H}_L(\tilde{w})\tilde{\upsilon}$. Moreover, we can recover from $\tilde{w}$ the laws of the system that generated $\tilde{w}$. This leads us to the following definition.

**Definition 3.** $\tilde{w} \in \mathfrak{B}$ is *sufficiently informative* about $\mathfrak{B}$ if colspan$(\mathfrak{H}_L(\tilde{w})) = \mathfrak{B}_{|[1,L]}$.

### 2.2. Interconnection

We introduce some relevant concepts of control by interconnection, see [7,8]. Let $c$ and $w$ denote the *control* and the *to-be-controlled* variables, respectively. Let the to-be-controlled *plant full behavior* be defined by

$$\mathcal{P}_{full} := \{(w, c) : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} \times \mathbb{R}^{\mathtt{c}} \mid (w, c) \text{ satisfies the plant equations}\}$$

and the *plant manifest behavior* by

$$\pi_w(\mathcal{P}_{full}) = \mathcal{P} := \{w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} | \exists\, c \text{ s.t. } (w, c) \in \mathcal{P}_{full}\}.$$

Finally, let a controller acting on the control variables be described by the *control behavior*

$$\mathcal{C} := \{c : \mathbb{Z} \to \mathbb{R}^{\mathtt{c}} | c \text{ satisfies the controller equations}\}.$$

The interconnection of the plant and the controller through the control variables denoted by $\mathcal{P}_{full} \wedge_c \mathcal{C}$ is defined by the *full controlled behavior*,

$$\mathcal{K}_{full} := \{(w, c) : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} \times \mathbb{R}^{\mathtt{c}} | (w, c) \in \mathcal{P}_{full} \text{ and } c \in \mathcal{C}\}.$$

$\mathcal{K}_{full}$ induces a *manifest controlled behavior* defined by

$$\mathcal{K} := \{w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} | \exists\, c \in \mathcal{C} \text{ s.t. } (w, c) \in \mathcal{P}_{full}\} = \pi_w(\mathcal{P}_{full} \wedge_c \mathcal{C}).$$

$\mathcal{K}$ is said to be *implementable* with respect to $\mathcal{P}_{full}$ if there exists a controller $\mathcal{C}$ such that $\mathcal{K} = \mathcal{P}_{full} \wedge_c \mathcal{C}$. It has been proven in Theorem 1 of [13] that $\mathcal{C}$ such that $\mathcal{K} = \mathcal{P}_{full} \wedge_c \mathcal{C}$ exists if and only if $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$, where $\mathcal{N} := \{w \in \mathcal{P} | (w, 0) \in \mathcal{P}_{full}\}$. In this paper we are interested in the case when $\mathcal{N} = 0$. Hence, we assume that any sub-behavior of $\mathcal{P}$ is implementable. Moreover, a special interconnection case of interest, called *full interconnection* arises when $w = c$. Under full interconnection the interconnection of the plant and the controller through $w$ is denoted by $\mathcal{P} \wedge_w \mathcal{C}$ and induces a controlled behavior defined by $\mathcal{K} := \{w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} | w \in \mathcal{P} \text{ and } w \in \mathcal{C}\}$.

## 3. Problem statements

In this section, we define formally the problems solved in this paper. Let the to-be-controlled system full behavior be

$$\mathcal{P}_{full} = \{(w, c) | R_1(\sigma)w = M_1(\sigma)c\} \tag{1}$$