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Controllability of linear passive network behaviors

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ABSTRACT

Classical RLC realization procedures (e.g. Bott–Duffin) result in networks with uncontrollable driving-point behaviors. With this motivation, we use the behavioral framework of Jan Willems to provide a rigorous analysis of RLC networks and passive behaviors. We show that the driving-point behavior of a general RLC network is stabilizable, and controllable if the network contains only two types of elements. In contrast, we show that the full behavior of the RLC network need not be stabilizable, but is marginally stabilizable. These results allow us to formalize the phasor approach to RLC networks using the notion of sinusoidal trajectories, and to address an assumption of conventional phasor analysis. Finally, we show that any passive behavior with a hybrid representation is stabilizable. This paper relies substantially on the fundamental work of our late friend and colleague Jan Willems to whom the paper is dedicated.

1. Introduction

Traditionally, questions of synthesis for passive networks have been framed in terms of achieving a given transfer function at the driving-point terminals. Brune's seminal contribution [1] was to show that a rational function can be realized as the impedance of a 2-terminal passive network if and only if it is positive-real. Subsequently, the Bott–Duffin procedure [2], and the reactance extraction scheme of Youla and Tissi [3], demonstrated that this result also holds for (two-terminal) RLC networks and multi-port passive networks, respectively. It has long been observed that the Bott–Duffin procedure, and its variants [4–7], generate RLC networks which contain a greater number of energy storage elements than the McMillan degree of the transfer function to be realized. Nevertheless, it has recently been established that, for the realization of certain transfer functions, these networks actually contain the least possible number of energy storage elements [8,9]. These facts motivate a fundamental treatment of the analysis of passive networks in a manner appropriate to the study of this apparent non-minimality. The behavioral approach and the dissipativity concept of Jan Willems are ideally suited to this task.

One significant contribution of the behavioral approach is a representation-free definition of the concepts 'controllability', 'stabilizability', and 'marginal stabilizability' (see [10, pp. 70–71] and Section 2 of this paper). As emphasized in [11, Section 8.2.3], the transfer function of a system is only sufficient for determining its behavior when that behavior is controllable. Indeed, despite the aforementioned necessary and sufficient conditions on the transfer functions of passive networks, it is still not known what are the necessary conditions for a (not necessarily controllable) behavior to be realized as the driving-point behavior of a passive network [12, Section 12]. In this paper, we will derive additional necessary conditions pertaining to the stabilizability of the driving-point behavior and full behavior of an RLC network, and to the controllability of the driving-point behavior of any network which contains only two types of elements (an LC, RC, or RL network).

The structure of the paper, and the key contributions, are as follows: We begin with some notation and preliminaries in Section 2. Throughout Sections 3–6, our focus is on RLC networks, which we analyze using a combination of the behavioral framework of Jan Willems with graph theory results from [13]. The approach in these sections is principally algebraic and exploits the correspondence between linear time-invariant differential systems and polynomial modules in the manner of Willems [11,10]. With these tools, we show that the driving-point behavior of an RLC network is necessarily stabilizable (Theorem 2), and controllable when the network contains only two types of elements (Theorem 3). In contrast, we show that the full behavior of an RLC network need not be stabilizable, but is necessarily marginally stabilizable (Theorem 4). With these results, we formalize the phasor analysis of RLC networks using the idea of sinusoidal trajectories, and address an assumption of conventional phasor analysis (Theorem 5). Then, in Section 7, we investigate the controllability of the driving-point and full behavior of the Bott–Duffin networks. Finally, in Section 8, we adopt a different approach aligned with the

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dissipativity concept of Willems, particularly the representation-free definition proposed in [12, Section 8], in order to study passive behaviors in greater generality. In particular, we show that any passive behavior with a hybrid representation is stabilizable. This result is applicable to the terminal behavior of any network comprising an interconnection of the usual passive elements (resistor, capacitor, inductor, transformer, gyrator).

2. Notation and preliminaries

In this paper, $\mathbb{R}$ (resp. $\mathbb{C}$, $\mathbb{Z}$) will denote the real (resp. complex, integer) numbers. For $z \in \mathbb{C}$, $\Re(z)$ denotes its real-part. $\mathbb{C}_+$ (resp. $\mathbb{C}_-$, $\bar{\mathbb{C}}_+$, $\bar{\mathbb{C}}_-$) denotes the open right half plane (resp. open left half plane, closed right half plane, closed left half plane). $\mathbb{R}[s]$ and $\mathbb{R}(s)$ will denote the space of polynomials and rational functions with real coefficients, respectively. $\mathbb{R}_p(s)$ will denote the proper (i.e. bounded at infinity) rational functions, and $\mathcal{RH}_\infty$ will denote the subspace of $\mathbb{R}_p(s)$ containing those functions which are analytic in $\bar{\mathbb{C}}_+$. Let $\mathbb{F}$ be one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}[s]$, $\mathbb{R}(s)$, or $\mathbb{R}_p(s)$. Then $\mathbb{F}^{m\times n}$ (resp. $\mathbb{F}^m$) denotes the matrices with m rows and n columns (resp. vectors with m rows) whose entries are from $\mathbb{F}$, and we write $\mathbb{F}^{\bullet\times\bullet}$ (resp. $\mathbb{F}^\bullet$) when the dimensions are immaterial. M^* will denote the Hermitian transpose of $M \in \mathbb{C}^{m\times n}$. For $D \in \mathbb{C}^{m\times m}$, $D \geq 0$ (resp. $D \leq 0$) indicates that D is positive (resp. negative) semidefinite. I_m will denote the identity matrix with m rows and m columns, and the dimension will occasionally be omitted when it is clear from the context. Finally, $\text{diag}\begin{pmatrix} x_1 & \cdots & x_m \end{pmatrix}$ will denote the diagonal matrix whose diagonal entries are $x_1, \ldots, x_m$, and $\text{col}\begin{pmatrix} A_1 & \cdots & A_m \end{pmatrix}$ will denote the block column matrix $\text{col}\begin{pmatrix} A_1 & \cdots & A_m \end{pmatrix} = \begin{bmatrix} A_1^T & \cdots & A_m^T \end{bmatrix}^T$.

We say that $H \in \mathbb{R}^{n\times n}(s)$ is positive-real (PR) if (i) H is analytic in $\mathbb{C}_+$, and (ii) $H(\xi)^* + H(\xi) \geq 0$ for all $\xi \in \mathbb{C}_+$. Here, (ii) is equivalent to (iia) $H(j\omega)^* + H(j\omega) \geq 0$ for all $\omega \in \mathbb{R}$ with $j\omega$ not a pole of any element of H, and (iib) any poles of H on $j\mathbb{R} \cup \infty$ are simple and have a positive semidefinite residue matrix [14, Thm 2.7.2].

Throughout this paper, we will consider linear time-invariant differential behaviors in the sense of [11], which we will frequently describe as the kernel of a differential operator $R(\frac{d}{dt})$ for some $R \in \mathbb{R}^{\bullet\times\bullet}[\xi]$. We refer to a particular element of the behavior as a trajectory. As in [11], we will consider behaviors to comprise trajectories which are locally integrable, i.e. $\mathcal{B} := \{\mathbf{b} \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^\bullet) \mid R(\frac{d}{dt})\mathbf{b} = 0\}$. Here, differentiation is interpreted in a weak sense, and we identify any two locally integrable functions which are equal except on a set of measure zero (see [11, Section 2.3.2]). We will denote the subset of $\mathcal{B}$ comprising the infinitely differentiable trajectories by $\mathcal{B} \cap \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) := \{\mathbf{b} \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \mid R(\frac{d}{dt})\mathbf{b} = 0\}$, and we note that any trajectory in $\mathcal{B} \cap \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ is also a solution to $R(\frac{d}{dt})\mathbf{b} = 0$ in the usual sense [11, Theorem 2.3.11].

If $\tilde{R} = UR$ for some unimodular U, then the sets of locally integrable functions in the kernels of $R(\frac{d}{dt})$ and $\tilde{R}(\frac{d}{dt})$ are identical [11, Theorem 2.5.4]. In particular, this enables the elimination of variables from a behavior $\mathcal{B} := \{\mathbf{b} \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^\bullet) \mid R(\frac{d}{dt})\mathbf{b} = 0\}$. Suppose $\mathbf{b}$ is partitioned as $\mathbf{b} =: \text{col}\begin{pmatrix} \mathbf{d} & \mathbf{r} \end{pmatrix}$, and we wish to eliminate $\mathbf{r}$ from $R(\frac{d}{dt})\mathbf{b} =: R_1(\frac{d}{dt})\mathbf{d} + R_2(\frac{d}{dt})\mathbf{r} = 0$. By [11, Theorem 2.5.23], there exists a unimodular $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}^T$ such that

$$\begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} R_1 = \begin{bmatrix} D_{1,1} \\ D_{2,1} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} R_2 = \begin{bmatrix} 0 \\ D_{2,2} \end{bmatrix}, \tag{1}$$

and where $D_{2,2}(\xi)$ has full row rank (equal to the rank of $R_2(\xi)$) for almost all $\xi \in \mathbb{C}$. From [15], then the behavior $\mathcal{B}_d := \{\mathbf{d} \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^\bullet) \mid \exists \mathbf{r} \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^\bullet)$ which satisfies $R_1(\frac{d}{dt})\mathbf{d} + R_2(\frac{d}{dt})\mathbf{r} = 0\}$ is equal to the set of solutions to $D_{1,1}(\frac{d}{dt})\mathbf{d} = 0$ which satisfy certain smoothness conditions. In some cases $\mathcal{B}_d = \{\mathbf{d} \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^\bullet) \mid D_{1,1}(\frac{d}{dt})\mathbf{d} = 0\}$, in which case we call r *properly eliminable* (see [15, Theorems 2.5 and 2.8], which contain criteria for proper eliminability).

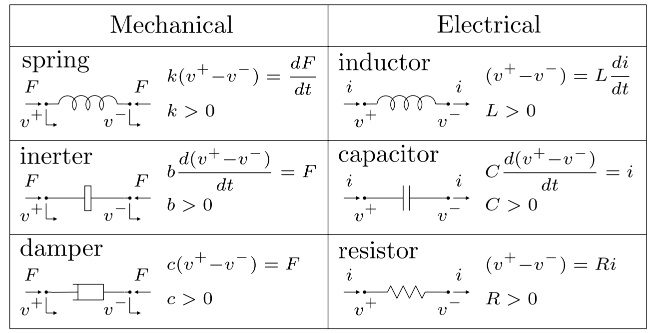

Fig. 1. Passive electrical and mechanical elements.

A behavior $\mathcal{B}$ is called *controllable* if for any two trajectories $\mathbf{b}_1, \mathbf{b}_2 \in \mathcal{B}$, there exists a $t_1 \geq 0$ and a $\mathbf{b} \in \mathcal{B}$ which satisfies $\mathbf{b}(t) = \mathbf{b}_1(t)$ for all $t \leq 0$ and $\mathbf{b}(t) = \mathbf{b}_2(t)$ for all $t \geq t_1$ [11, Definition 5.2.2]. It is called *stabilizable* if for every $\mathbf{b}_1 \in \mathcal{B}$, there exists a $\mathbf{b} \in \mathcal{B}$ which satisfies $\mathbf{b}(t) = \mathbf{b}_1(t)$ for all $t \leq 0$ and $\lim_{t\to\infty} \mathbf{b}(t) = 0$; and *marginally stabilizable* if for every $\mathbf{b}_1 \in \mathcal{B}$, there exists a $t_1 \geq 0$ and a $\mathbf{b} \in \mathcal{B}$ which satisfies $\mathbf{b}(t) = \mathbf{b}_1(t)$ for all $t \leq 0$ and $\mathbf{b}(t)$ is bounded in $t \geq t_1$. From [11, Thms 5.2.10 and 5.2.30], whenever $R \in \mathbb{R}^{\bullet\times\bullet}[s]$, then the behavior $\mathcal{B} := \{\mathbf{b} \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^\bullet) \mid R(\frac{d}{dt})\mathbf{b} = 0\}$ is controllable (resp. stabilizable) if and only if the rank of $R(\xi)$ is the same for all $\xi \in \mathbb{C}$ (resp. $\xi \in \bar{\mathbb{C}}_+$).

Further relevant material is provided as footnotes 1–5.

3. RLC networks and behaviors

In this section, we present explicit and parsimonious descriptions of the full behavior and the driving-point behavior of a given RLC network.

We define resistors, inductors and capacitors as the idealized elements shown in Fig. 1. Each such element is associated with a current $i \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R})$ through the element and a voltage $v := v^+ - v^- \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R})$ across the element which are constrained to satisfy the corresponding differential equation given in that figure. We remark that, by identifying force with current and velocity with voltage, there is a direct analogy between RLC networks and mechanical networks comprising dampers, springs, and inerters (see Fig. 1). Consequently, the conclusions of this paper are equally applicable to the analysis of such mechanical networks.

In Sections 3–6, we restrict attention to RLC 'one-port' networks. Any such network N has the structure of a connected, oriented graph[1] (hereafter referred to as a graph) with vertices $x_1, \ldots, x_n$ and edges $y_1, \ldots, y_m$, and with two designated *external* vertices corresponding to the two terminals which constitute the port of N. Each edge y_k represents an element N_k ($k = 1, \ldots, m$) which is either a resistor, an inductor, or a capacitor. If all edges correspond to either inductors or capacitors (resp. resistors or capacitors, resistors or inductors) then we call N an LC (resp. RC, RL) network. The edges are oriented so that the current i_k through N_k is from tail to head, and $\int_{t_0}^{t_1} i_k(t)v_k(t)dt$ is the energy supplied to N_k

[1] By a *graph* we mean an ordered pair (V, E) where V is a set $\{x_1, \ldots, x_n\}$ whose elements are called *vertices* and E is a set $\{y_1, \ldots, y_q\}$ of unordered pairs of vertices called *edges*, i.e. $y_k = (x_{k_1}, x_{k_2})$, $k = 1, \ldots, q$ [13]. Our definition of connected follows [13]. In contrast to [13], we allow several edges to join the same two vertices. A graph is called oriented when each edge has one of its vertices arbitrarily assigned as a head vertex and the other as a tail vertex.

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