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Constrained controllability of fractional linear systems with delays in control

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ABSTRACT

The paper presents finite-dimensional dynamical control systems described by linear fractional-order state equations with multiple delays in control. The constrained controls are considered. The set of admissible control values is assumed to be a compact set containing 0, a convex and compact set containing 0 in its interior, a cone with vertex at zero and a nonempty interior, or a convex and close cone with nonempty interior and vertex at zero. The definitions of various types of constrained controllability of the linear fractional systems with delays in control are formulated. New necessary and sufficient conditions for constrained relative controllability of fractional-order control systems with delayed controls are established and proved. Numerical examples are provided to illustrate the obtained theoretical results.

1. Introduction

The controllability of dynamical systems is one of the most important issues in control theory. In general, the controllability means that it is possible to steer a control system from an initial state into a final state with the aid of admissible controls. Many different controllability definitions have been formulated in literature, which depend on both a class of control systems and a set of admissible controls. A review of recently analyzed controllability problems for a wide class of dynamical systems is presented in [1].

In recent decades, the issues addressed in papers concerning the controllability of dynamical control systems focused on systems defined by fractional-order differential equations. Fractional-order models have proven to describe the behavior of many real-life processes more accurately. Control systems modeled with the use of fractional differential equations occur, among others, in mechanical, biological and chemical problems. Detailed discussions of fractional differential equations and their practical applications can be found in the following monographs: [2–8].

The controllability of linear fractional-order control systems has been studied in many monographs and papers. The controllability of discrete-time fractional systems is studied in [9–11], positive fractional discrete-time systems are discussed in [12], positive fractional linear systems, both discrete- and continuous-time, are presented in [13] and [14]. The controllability of continuous time linear fractional systems is studied, among others, in [15–21].

In many processes, future states depend on both the present state and past states of a system. This means that models describing the processes involve delays in state or in control. If we have delays in the input function, we deal with control systems with delayed controls. In view of the apparent large number of mathematical models which describe dynamical systems with delays in control, solving controllability problems for such systems is of particular importance. The controllability problems for linear continuous-time fractional systems with delayed control were analyzed in [13,14,22–24]. It should be noted, however, that the majority of papers on linear fractional systems controllability address controllability issues for unconstrained controls. The works on the controllability of fractional systems with bounded inputs are [25] for fractional positive discrete-time linear systems, [26] for fractional positive continuous-time linear systems, [27] for continuous-time linear fractional systems, [28] for systems represented by fractional integrodifferential equations, and [29] for linear fractional systems with h-difference fractional operator of the Caputo type. It needs to be clearly stated that, in practice, controls are not completely unconstrained. The set of admissible controls is usually bounded in some way.

Fractional differential equations occur, for example, in mathematical models of biological and biochemical models such as: cancer models [30], population growth models [31,32], models of

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migration of interacting agents [33]. In the majority of biological models, positive controls are considered.

The aim of the paper is to give new criteria for the constrained controllability of continuous-time fractional-order control systems with multiple delays in control. In [27] some constrained controllability criteria have been formulated. The majority of the criteria are based on the so-called supporting function. In the present paper we propose some criteria based on the stability of fractional systems with delays in control and several "rank-type" controllability criteria. The proposed new criteria are easy to verify, which is shown by the illustrative examples. However, the most important difference with the paper [27] is the modification of state space description of the fractional-order system that takes into account the so-called internal states, and allows to consider the behavior of the system in a convenient way.

The paper is organized in the following manner. Section 2 recalls some preliminary definitions, formulas and notations. In Section 3, the mathematical model of the considered fractional systems with point delays in control is presented. The formula for a solution of the discussed systems is presented and some definitions of constrained relative controllability of the system are formulated. Constraints are imposed on control values. The main results of the paper, contained in Section 4, are the criteria for constrained relative controllability of the considered fractional system with delayed controls in the following cases: a set of admissible control values U is a convex and compact set containing 0 in its interior, it is a closed and convex cone with nonempty interior and vertex at zero, it is a cone with vertex at zero and nonempty interior in the space $\mathbb{R}^m$, which implies the case of positive controls. Proofs of the theorems are presented. In Section 5, the theoretical results are illustrated with numerical examples. Finally, concluding remarks are included in Section 6.

2. Preliminaries and notation

Before we present the system description, we recall some notions concerning fractional-order differential equations. Fractional-order differentiation is the generalization of the integer-order one. There are several definitions of fractional-order derivatives, among others: the Grünwald–Letnikov, the Riemann–Liouville, the Caputo fractional derivatives [5]. In this paper we use Caputo's fractional derivatives.

The Caputo fractional derivative of order α ($n < \alpha < n+1$, $n \in \mathbb{N}$) for a differentiable function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined as

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha+1)} \int_0^t \frac{f^{(n+1)}(\tau)}{(t-\tau)^{\alpha-n}}\, d\tau,$$

where Γ is the gamma function.

It is obvious that for $\alpha \to n$, Caputo's derivative approaches the nth order conventional derivative of f, that is $\lim_{\alpha\to n} {}^{C}D^{\alpha} f(t) = f^{(n)}(t)$.

Based on the definition of the Mittag-Leffler function [5,13]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)},\ z \in \mathbb{C},\ \alpha > 0,\ \beta > 0,$$

for an arbitrary nth order square matrix A we can give the formula for a *pseudo-transition matrix* $\Phi_0(t)$ of the linear fractional system $^{C}D^{\alpha} x(t) = Ax(t)$ [8,13]

$$\Phi_0(t) = E_{\alpha,1}(At^{\alpha}) = \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma(k\alpha+1)}.$$

Moreover, let

$$\Phi(t) = t^{\alpha-1} E_{\alpha,\alpha}(At^{\alpha}) = t^{\alpha-1} \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma((k+1)\alpha)}.$$

For $\alpha = 1$ we obtain the classical transition matrix of ordinary differential equations

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = e^{At}.$$

Therefore the pseudo-transition matrix $\Phi_0(t)$ is also called the matrix α-exponential function and is denoted by $\Phi_0(t) = e_{\alpha}^{At}$ [6,13].

The following notation is used throughout the paper. Let $L^2([0,\infty), \mathbb{R}^m)$ denote the Hilbert spaces of square integrable functions with values in $\mathbb{R}^m$, and $L^2_{loc}([0,\infty), \mathbb{R}^m)$ denote the linear space of locally square integrable functions with values in $\mathbb{R}^m$ whereas $L_{\infty}([0,T], U)$ means the Banach space of functions bounded almost everywhere, defined on $[0,T]$ with values in U.

3. System description

In the paper the linear control systems with multiple delays in control described by the following fractional-order differential state equation are studied.

$$^{C}D^{\alpha} x(t) = A\,x(t) + \sum_{i=0}^{M} B_i\, u(t-h_i) \tag{1}$$

for $t \geq 0$ and $0 < \alpha < 1$, where

- $x(t) \in \mathbb{R}^n$ is a state vector,
- $u \in L^2_{loc}([0,\infty), \mathbb{R}^m)$ is a control,
- A is a $(n \times n)$-matrix with real elements,
- B_i are $(n \times m)$-matrices with real elements for $i = 0, 1, \ldots, M$,
- $h_i \in \mathbb{R}$, $i = 1, 2, \ldots, M$ are constant point delays in control that satisfy the following inequalities

  $$0 = h_0 < h_1 < \cdots < h_i < \cdots < h_{M-1} < h_M.$$

The works [34–37] on initialization in fractional-order systems show that initial conditions are not taken into account in the same way when Riemann–Liouville or Caputo definitions of fractional derivatives are considered. Moreover, in [38–43] it has been shown that the Caputo definition does not permit to take into account initial conditions in a coherent way, because it does not permit to consider in a convenient way the physical behavior of the system.

The state vector $x(t)$ of the system (1) is called pseudo state (see: [43,44]), because it is the following weight integral

$$x(t) = \int_0^{\infty} \mu_{\alpha}(\omega) z_C(\omega, t) d\omega,$$

where the internal state $z_C(\omega, t)$ is the true state of the system and $\mu_{\alpha}(\omega) = \frac{\sin \alpha\pi}{\pi}\, \omega^{-\alpha}$ [45].

Therefore, we introduce a supplementary term corresponding to the internal state of a specific fractional integrator [45]. For the Caputo derivative we have

$$^{C}D^{\alpha} x(t) = I_{1-\alpha}\Big(\frac{dx(t)}{dt}\Big),$$

where $I_{1-\alpha}(\cdot)$ represents the fractional integration with order $1-\alpha$. Let $z_C(\omega, t)$ be the internal state of the integrator $I_{1-\alpha}(\cdot)$. Then the revised Laplace transform of $^{C}D^{\alpha} x(t)$ takes the form

$$\mathcal{L}\Big[{}^{C}D^{\alpha} x(t)\Big] = s^{\alpha} \mathcal{L}[x(t)] - s^{\alpha-1} x(0) + \int_0^{\infty} \frac{\mu_{1-\alpha}(\omega) z_C(\omega, 0)}{s+\omega}\, d\omega.$$

Hence, the initial conditions of Caputo's derivative are x(0) and $z_C(\omega, 0)$.

Remark 3.1. In [46,47] it was shown that fractional models are physically inconsistent models. However, the fractional-order models have similar properties as systems with delays. In both,