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External stability of switching control systems☆

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ABSTRACT

This paper investigates external stability (defined by the normal L_2 norm) of switching control systems. It proposes definitions of the maximum, minimum dwell time for switching systems and then derives an important relation between the number of switchings and the maximum, minimum dwell time. Applying this relation, it establishes criteria on external stability of switching control systems consisting of Hurwitz stable subsystems. Furthermore, a switching law is proposed, and is proven to be realizable. Given this proposed switching law and applying the previous derived relation, the switching control systems comprised of both Hurwitz stable and Hurwitz unstable subsystems are proved to be externally stable.

1. Introduction

For a control system, if we think of $u \in L_2$ as a "well behaved" input, the question to ask is whether the output y will be "well behaved" in the sense that $y \in L_2$ [1]. If any "well behaved" input generates a "well-behaved" output, then the control system will be defined as a stable system [1]. This type of stability is also called external stability [2]. External stability plays a special role in system analysis because it is natural to work with square-integrable signals which can be viewed as finite-energy signals [1]. If one thinks of $u(t)$ as current or voltage, then $u^T(t)u(t)$ is proportional to the instantaneous power of the signal, and its integral over all time is a measure of the energy content of the signal. Certainly, others like random signals may be also usable [3]. Switching systems, consisting of a family of subsystems and a switching law that determines which subsystem is active during a certain time interval [4], have been used as an important framework for externally stable (or related) control in recent years [5–8]. In [5], authors investigated the weighted disturbance attenuation properties (the weighted L_2 norm of output is bounded by the L_2 norm of disturbance input) of switched systems. However, 'weighted' changes the original meaning and no explicit switching law is proposed for those systems consisting of both Hurwitz stable and Hurwitz unstable subsystems. In [6,7], the weighted input–output relation following from [5] was adopted. In [8], a normal L_2 relation between input and output was studied but the average dwell time could not be directly substituted into the integral in (36) of [8]. Up to now, the external stability of switching control systems still remains open. We shall investigate it in this paper.

2. Preliminaries

Consider the following switching control system

$$
\begin{aligned}
\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t),\\
y(t) &= C_{\sigma(t)}x(t) + E_{\sigma(t)}u(t),\\
x(t_0) &= x_0,
\end{aligned}
\tag{1}
$$

where $x(t) \in R^n$, $u(t) \in R^m$ and $y(t) \in R^p$ denote the state, input and output, respectively. The number of state variables n is called the order of the control system. $A_{\sigma(t)}, B_{\sigma(t)}, C_{\sigma(t)}$ and $E_{\sigma(t)}$ are known matrices with appropriate dimensions, where $\sigma(t)$ is the switching signal, which takes values from an index set $\mathscr{P} = \{1, 2, \ldots\}$ and denotes the number of the active subsystem at t, i.e. $\sigma(t) = i \in \mathscr{P}$ means the subsystem i is active at t. Corresponding to $\sigma(t)$, there is a sequence of time instants $t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$, where t_k, $k = 1, 2, \ldots$, is the switching instant and $t_1 > t_0$. Physically, $A_{\sigma(t)}$ describes the internal dynamics; $B_{\sigma(t)}$ the effect of the controlled input on the state; and $C_{\sigma(t)}$, $E_{\sigma(t)}$ describe the sensors.

Definition 2.1 (*[2]*). A system is externally stable (or L_2 stable) if, for every $u \in L_2([0, \infty); R^m)$, the output y is in $L_2([0, \infty); R^p)$. (Here y is the zero-state output.)

Definition 2.2 (*[2]*). The L_2 gain of an externally stable system is $\gamma = \sup_{u\in L_2, u\neq\theta} \frac{\|y\|_{L_2}}{\|u\|_{L_2}}$. (Here θ indicates the zero function.) The L_2 gain is the maximum ratio of $\|y\|_{L_2}/\|u\|_{L_2}$.

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Definition 2.3 ([9]). For each $t \geq \tau \geq 0$, let $N_\sigma(t,\tau)$ denote the number of discontinuities of σ over (τ, t),

$$N_\sigma(t,\tau) \leq N_0 + \frac{t-\tau}{T_a}, \tag{2}$$

for given N_0, T_a. The constant T_a is called the average dwell time and N_0 the chatter bound.

The "average dwell time" is an important concept for switching systems, and has been used in many control problems, e.g. disturbance attenuation [5], L_2 gain analysis [6] and weighted H_∞ model reduction [7]. However, because (2) is a one-direction inequality, a weighted term $e^{-\alpha s}$ cannot be canceled when applying (2) to prove the L_2 norm inequality for these control problems. So in these literatures, authors had to adopt the weighted L_2 norm. We shall demonstrate this dilemma in Remark 3.1. In order to overcome this difficulty (derive a two-direction inequality with regard to $N_\sigma(t,\tau)$), we propose definitions of the maximum, minimum dwell time as follows.

Definition 2.4. The maximum dwell time of a switching system is $T_{max} = \sup_{k=1,2,\ldots}(t_k - t_{k-1})$; the minimum dwell time of a switching system is $T_{min} = \inf_{k=1,2,\ldots}(t_k - t_{k-1})$.

Note that the definition above is only for those systems with finite maximum dwell time and nonzero minimum dwell time.

Lemma 2.1. *For any* $t \geq \tau \geq t_0$,

$$\frac{t-\tau}{T_{max}} - 1 \leq N_\sigma(t,\tau) \leq \frac{t-\tau}{T_{min}} + 1, \tag{3}$$

where T_{max}, T_{min} *are the maximum and minimum dwell time of system* (1).

Proof. Without loss of generality, assume that $t \in (t_{k-1}, t_k]$, $k = 1, 2, \ldots$, then prove through two different cases: $t \in (t_{k-1}, t_k)$ and $t = t_k$.

Case $t \in (t_{k-1}, t_k)$: $\tau \in (t_{i-2}, t_{i-1})$, $\tau = t_{i-1}$ or $\tau \in (t_{k-1}, t]$, $i = 2, 3, \ldots, k$. If $\tau \in (t_{i-2}, t_{i-1})$, then $N_\sigma(t,\tau) = k-i+1$, $t-\tau \leq T_{max}[N_\sigma(t,\tau)+1]$ and $\frac{t-\tau}{T_{min}} \geq N_\sigma(t,\tau) - 1$ such that (3) holds. If $\tau = t_{i-1}$, then $N_\sigma(t,\tau) = k-i$, $t-\tau \leq T_{max}[N_\sigma(t,\tau)+1]$ and $\frac{t-\tau}{T_{min}} \geq N_\sigma(t,\tau)$ such that (3) holds. If $\tau \in (t_{k-1}, t]$, $N_\sigma(t,\tau) = 0$, $t-\tau \leq T_{max}$ and $\frac{t-\tau}{T_{min}} \geq 0$ such that (3) holds.

Case $t = t_k$: $\tau \in (t_{i-2}, t_{i-1})$, $\tau = t_{i-1}$, $\tau \in (t_{k-1}, t_k)$, or $\tau = t_k$, $i = 2, 3, \ldots, k$. If $\tau \in (t_{i-2}, t_{i-1})$, $N_\sigma(t,\tau) = k-i+2$, $t-\tau \leq T_{max}N_\sigma(t,\tau)$ and $\frac{t-\tau}{T_{min}} \geq N_\sigma(t,\tau) - 1$ such that (3) holds. If $\tau = t_{i-1}$, then $N_\sigma(t,\tau) = k-i+1$, $t-\tau \leq T_{max}N_\sigma(t,\tau)$ and $\frac{t-\tau}{T_{min}} \geq N_\sigma(t,\tau)$ such that (3) holds. If $\tau \in (t_{k-1}, t_k)$, $N_\sigma(t,\tau) = 1$, $t-\tau \leq T_{max}$ and $\frac{t-\tau}{T_{min}} \geq 0$ such that (3) holds. If $\tau = t_k$, then $N_\sigma(t,\tau) = 1$ and $t-\tau = 0$ such that (3) holds.

Therefore, (3) always holds. □

As we see, (3) is two-direction, and will be applied to prove the normal (instead of weighted) L_2 norm inequality for external stability, see (16)–(20).

3. Main results

In this section, we first prove the external stability of switching control systems consisting of Hurwitz stable subsystems by using relation (3). In order to show the external stability of switching control systems containing Hurwitz unstable subsystems, we propose a switching law and prove it to be realizable. Given this switching law and applying the relation (3), we finally show the switching control systems comprised of both Hurwitz stable and Hurwitz unstable subsystems to be externally stable.

Theorem 3.1. *For the given scalars* $\alpha > 0$, $\mu > 0$ *and the numbers of any two consecutive subsystems* $i, j \in \mathscr{P}$ *(*σ *jumps from* j *to* i*), the control system* (1) *is externally stable with a* L_2 *gain* γ*, if there exist real* $n \times n$ *matrices* P_i, $P_j > 0$ *such that*

$$\begin{bmatrix} A_i^T P_i + P_i A_i + \alpha P_i & P_i B_i & C_i^T \\ * & -\Gamma_\mu^2 I & E_i^T \\ * & * & -I \end{bmatrix} < 0, \tag{4}$$

$$P_i \leq \mu P_j \tag{5}$$

and

$$\alpha > \frac{\ln \mu}{T_{min}}, \tag{6}$$

where $\Gamma_\mu = \frac{\gamma}{\mu}\sqrt{\frac{\alpha - \ln\mu/T_{min}}{\alpha - \ln\mu/T_{max}}}$ *if* $\mu > 1$; $\Gamma_\mu = \gamma$ *if* $\mu = 1$; $\Gamma_\mu = \gamma\mu\sqrt{\frac{\alpha - \ln\mu/T_{max}}{\alpha - \ln\mu/T_{min}}}$ *if* $0 < \mu < 1$, T_{max}, T_{min} *are the maximum and minimum dwell time of system* (1)*, respectively.*

Proof. Consider the following piecewise quadratic Lyapunov function candidate

$$V_{\sigma(t)}(x) = x^T P_{\sigma(t)} x. \tag{7}$$

Suppose $\sigma(t) = i$ for $t \in [t_{k-1}, t_k)$, then the derivative of $V_{\sigma(t)}(x)$ with respect to t along the trajectory of (1) on $[t_{k-1}, t_k)$ is

$$\begin{aligned} \dot{V}_{\sigma(t)}(t) &= \dot{V}_i(t) \\ &= x^T(t)(A_i^T P_i + P_i A_i)x(t) + u^T(t)B_i^T P_i x(t) \\ &\quad + x^T(t)P_i B_i u(t). \end{aligned} \tag{8}$$

Let $\Delta(t) = y^T(t)y(t) - \Gamma_\mu^2 u^T(t)u(t)$, then we have

$$\begin{aligned} &\dot{V}_i(t) + \alpha V_i(t) + \Delta(t) \\ &= x^T(t)(A_i^T P_i + P_i A_i)x(t) + u^T(t)B_i^T P_i x(t) \\ &\quad + x^T(t)P_i B_i u(t) + \alpha x^T(t)P_i x(t) + x^T(t)C_i^T C_i x(t) \\ &\quad + x^T(t)C_i^T E_i u(t) + u^T(t)E_i^T C_i x(t) + u^T(t)E_i^T E_i u(t) \\ &\quad - \Gamma_\mu^2 u^T(t)u(t) \\ &= \eta(t)^T \begin{bmatrix} A_i^T P_i + P_i A_i + \alpha P_i + C_i^T C_i & P_i B_i + C_i^T E_i \\ * & E_i^T E_i - \Gamma_\mu^2 I \end{bmatrix} \eta(t), \end{aligned} \tag{9}$$

where $\eta(t) = [x^T(t), u^T(t)]^T$. Applying the Schur complement [10], (4) implies

$$\dot{V}_i(t) + \alpha V_i(t) + \Delta(t) \leq 0. \tag{10}$$

Since $x(t)$ is continuous on $[t_{k-1}, t_k)$ due to $u(t) \in L_2([0,\infty); R^m)$ ($u(t)$ is piecewise continuous), $V_i(t)$ is continuous on $[t_{k-1}, t_k)$. In view of the derivative of $V_i(t)$ in (8), $\dot{V}_i(t)$ is also piecewise continuous on $[t_{k-1}, t_k)$. Thus, integrating the inequality above, from t_{k-1} to t, $t \in [t_{k-1}, t_k)$, yields

$$V_i(t) \leq V_i(t_{k-1})e^{-\alpha(t-t_{k-1})} - \int_{t_{k-1}}^{t} \Delta(s)e^{-\alpha(t-s)}\,ds. \tag{11}$$

Since subsystem i and subsystem j are consecutive (σ jumps from j to i), it follows from (5) that

$$V_i(t_{k-1}) \leq \mu V_j(t_{k-1}^-). \tag{12}$$

Apply the technique in (2.7) of [5] as following

$$\begin{aligned} V_i(t) &\leq \mu V_j(t_{k-1}^-)e^{-\alpha(t-t_{k-1})} - \int_{t_{k-1}}^{t} \Delta(s)e^{-\alpha(t-s)}\,ds \\ &\leq \mu[V_j(t_{k-2})e^{-\alpha(t_{k-1}-t_{k-2})} - \int_{t_{k-2}}^{t_{k-1}} \Delta(s)e^{-\alpha(t_{k-1}-s)}\,ds]e^{-\alpha(t-t_{k-1})} \\ &\quad - \int_{t_{k-1}}^{t} \Delta(s)e^{-\alpha(t-s)}\,ds \\ &\leq \cdots \end{aligned}$$