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Finite-time stability theorems of homogeneous stochastic nonlinear systems ☆

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ABSTRACT

The purpose of this paper is to study finite-time stability of a class of homogeneous stochastic nonlinear systems modeled by stochastic differential equations. An existence result of weak solutions for stochastic differential equations with continuous coefficients is derived as a preparation for discussing stochastic nonlinear systems. Then a generalization of finite-time stochastic stability theorem is given. By using some properties of homogeneous functions and homogeneous vector fields, it is proved that a homogeneous stochastic nonlinear system is finite-time stable if its coefficients have negative degrees of homogeneity, and there exists a sufficiently smooth and homogeneous Lyapunov function such that the infinitesimal generator of the stochastic system acting on it is negative definite. In the case when the drift coefficient of a stochastic system is homogeneous and has a negative degree of homogeneity, it can be shown that the stochastic system is also finite-time stable under appropriate conditions. Two examples are provided as illustrations.

1. Introduction

Asymptotic stability in dynamical systems implies convergence of the system trajectories to an equilibrium state over the infinite horizon. However, in practice, it is desirable that a dynamical system possesses finite-time stability, that is, its trajectories converge to an equilibrium state in finite time. A rigorous framework for finite-time stability analysis of nonlinear dynamical systems was presented in [1] by using Lyapunov functions.

If a nonlinear dynamical system incorporates an additive stochastic disturbance input, and we take the stochastic disturbance input to be a Brownian motion, then a stochastic nonlinear system, described as an Itô stochastic differential equation, will be generated. The basic theory of stochastic integrals and stochastic differential equations driven by Brownian motion is given in [2], and the theory of asymptotic stability of stochastic systems is discussed in [3,4]. For a stochastic system, a question arises as to whether the stochastic counterpart of Theorem 4.2 in [1] can be obtained. A rigorous finite-time stability analysis for stochastic nonlinear systems was first made by Yin et al. [5] in the framework of Lyapunov functions. It also had been recognized that it is impossible to discuss finite-time stability for a stochastic nonlinear system with locally Lipschitz continuous coefficients. Therefore, a finite-time stable stochastic nonlinear system at least has one coefficient that does not satisfy the local Lipschitz condition. In this case, in general, the uniqueness of solutions in forward time for such a stochastic system cannot be ensured. Nevertheless, as was remarked in [6], the uniqueness of solutions is not necessary as opposed to the existence when studying the finite-time stability of a stochastic nonlinear system, since stability in probability implies that the origin is both an equilibrium point and an absorbing state (see Remark 2.2, [6]).

For deterministic nonlinear systems, homogeneous systems and homogeneous techniques have been used for the purposes of finite-time stability and stabilization [7,8], with the aid of the Lyapunov finite-time stability theorem in [1]. More particularly, it has been shown in [9] that a homogeneous system is finite-time stable if and only if it is asymptotically stable and has a negative degree of homogeneity. This motivates us to consider whether a homogeneous stochastic nonlinear system is also finite-time stable if it is asymptotically stable and has a negative degree of homogeneity. If not, what conditions can ensure that a homogeneous stochastic nonlinear system is finite-time stable at the origin? It seems that more conditions are needed when studying finite-time stability of stochastic systems due to the randomness and

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complexity of stochastic systems in comparison with deterministic systems.

The main purpose of this paper is to investigate the finite-time stability of homogeneous stochastic nonlinear systems by means of the Lyapunov stability theorem in [5] and the theory of homogeneous systems in [10–12]. It can be shown that a homogeneous stochastic nonlinear system is finite-time stable if its coefficients have negative degrees of homogeneity, and there exists a sufficiently smooth and homogeneous Lyapunov function such that the infinitesimal generator of the stochastic system acting on it is negative definite. When the drift coefficient of a stochastic nonlinear system is homogeneous, it is proved that the stochastic system is also finite-time stable at the origin under restrictive conditions on the diffusion coefficient and the Lyapunov function. This result also implies that it is possible to extend, at least partially, the result coming from [9] to the stochastic case.

The paper is organized as follows. Section 2 begins with two definitions of homogeneity for a function and a vector field. Some preliminary properties of homogeneous functions and homogeneous vector fields are given. Particularly, a link between the homogeneous 2-norm and the Euclidean norm is also established. Section 3 devotes to studying the finite-time stability of homogeneous stochastic nonlinear systems. As a preparation, it is proved that a stochastic nonlinear system with continuous coefficients has weak solutions if there exists a radially unbounded Lyapunov function V, which is twice continuously differentiable except possibly at each point in a negligible set U_0 and satisfies $\mathcal{L}V(x) < 0$ for any $x \in \mathbb{R}^n \setminus \{0\}$. Then a generalization of finite-time stochastic stability theorem is given. We first consider a class of homogeneous stochastic nonlinear systems whose coefficients have negative degrees of homogeneity. We show that such a stochastic system is finite-time stable at the origin if there exists a suitable homogeneous Lyapunov function. We further prove that a stochastic nonlinear system is finite-time stable if its drift coefficient has a negative degree of homogeneity and there is an appropriate homogeneous Lyapunov function. Also, two examples are provided as illustrations together with some remarks. Finally, some concluding remarks are given in Section 4.

2. Homogeneous functions and some properties

In this section we adopt the definitions of homogeneity for functions and vector fields (vector valued functions) given in [12] (see also, for example, [10,11]), where the dilation is used to define the homogeneity of a function or a vector field. The so-called dilation is a mapping of the form $\Delta_\varepsilon(x_1, \dots, x_n) = (\varepsilon^{r_1}x_1, \dots, \varepsilon^{r_n}x_n)$ with respect to $\varepsilon > 0$, where $x_1, \dots, x_n$ are fixed coordinates on $\mathbb{R}^n$ and $r_1, \dots, r_n$ are some positive real numbers. $(r_1, \dots, r_n)$ is also called the dilation weight. We denote by $C(\mathbb{R}^n, \mathbb{R}^p)$ the set of continuous (vector) functions from $\mathbb{R}^n$ onto $\mathbb{R}^p$, where p is an integer. Let $C^2(\mathbb{R}^n, \mathbb{R})$ denote the family of twice continuously differentiable functions from $\mathbb{R}^n$ onto $\mathbb{R}$.

Definition 2.1 (*Homogeneous Function*)**.** A function $V \in C(\mathbb{R}^n, \mathbb{R})$ is said to be homogeneous of degree $\tau \in \mathbb{R}$ with the dilation $\Delta_\varepsilon(x_1, \dots, x_n)$ if

$$V\big(\Delta_\varepsilon(x_1, \dots, x_n)\big) = \varepsilon^\tau V(x_1, \dots, x_n), \forall x \in \mathbb{R}^n \setminus \{0\}. \tag{2.1}$$

Definition 2.2. A vector field $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ is said to be homogeneous of degree $l \in \mathbb{R}$ with respect to the dilation $\Delta_\varepsilon(x_1, \dots, x_n)$ if the ith component f_i satisfies

$$\begin{aligned} &f_i\big(\Delta_\varepsilon(x_1, \dots, x_n)\big) \\ &\quad = \varepsilon^{l+r_i} f_i(x_1, \dots, x_n), \forall x \in \mathbb{R}^n \setminus \{0\}, i = 1, \dots, n. \end{aligned} \tag{2.2}$$

For any given dilation weight $(r_1, \dots, r_n)$ and constant $p \geq 1$, the homogeneous p-norm is defined as $\|x\|_{\Delta,p} = \big(\sum_{i=1}^n |x_i|^{p/r_i}\big)^{1/p}$, $\forall x \in \mathbb{R}^n$. If $p = 2$, $\|x\|_{\Delta,p}$ is written as $\|x\|_\Delta$ for simplicity. We denote by $\|x\|$ the usual Euclidian norm $\|x\| = \big(\sum_{i=1}^n x_i^2\big)^{1/2}$.

If differentiating both sides of Eq. (2.1) with respect to x_i and x_j, we can derive the following properties.

Lemma 2.1 (*[13]*)**.** *If $V \in C^2(\mathbb{R}^n, \mathbb{R})$ is a homogeneous function of degree $\tau \in \mathbb{R}$ with respect to a dilation weight $(r_1, \dots, r_n)$, then its partial derivatives $\partial V/\partial x_i$ and $\frac{\partial^2 V}{\partial x_i \partial x_j}$ are also homogeneous. More precisely,*

$$\frac{\partial V}{\partial x_i}\big(\Delta_\varepsilon(x_1, \dots, x_n)\big) = \varepsilon^{\tau - r_i} \frac{\partial V}{\partial x_i}\big(x_1, \dots, x_n\big), i = 1, \dots, n, \tag{2.3}$$

$$\begin{aligned} &\frac{\partial^2 V}{\partial x_i \partial x_j}\big(\Delta_\varepsilon(x_1, \dots, x_n)\big) \\ &\quad = \varepsilon^{\tau - r_i - r_j} \frac{\partial^2 V}{\partial x_i \partial x_j}\big(x_1, \dots, x_n\big), i, j = 1, \dots, n. \end{aligned} \tag{2.4}$$

The next lemma provides a link between the homogeneous 2-norm and the Euclidean norm. This property will play an important role in proving finite-time stability of homogeneous stochastic nonlinear systems.

Lemma 2.2. *Suppose $V \in C(\mathbb{R}^n, \mathbb{R})$ is positive definite and homogeneous with a degree $\tau > 0$ with respect to a dilation weight $(r_1, \dots, r_n)$. Then the following inequalities hold:*

$$V(x_1, \dots, x_n) \leq \bar{c}\, \|x\|_\Delta^\tau \leq \hat{\bar{c}}\Big[\|x\|^{\frac{\tau}{\bar{r}}} 1_{0 \leq \|x\| \leq 1} + \|x\|^{\frac{\tau}{\underline{r}}} 1_{\|x\| > 1}\Big], \tag{2.5}$$

and

$$V(x_1, \dots, x_n) \geq \underline{c}\, \|x\|_\Delta^\tau \geq \underline{\hat{c}}\Big[\|x\|^{\frac{\tau}{\underline{r}}} 1_{0 \leq \|x\| \leq 1} + \|x\|^{\frac{\tau}{\bar{r}}} 1_{\|x\| > 1}\Big], \tag{2.6}$$

where $\underline{r} = \min r_i$, $\bar{r} = \max r_i$, and $0 < \underline{\hat{c}} \leq \underline{c} \leq \bar{c} \leq \hat{\bar{c}}$ are some constants.

Proof. Without any loss of generality, we only need to prove (2.5) and (2.6) in the case of $x \in \mathbb{R}^n \setminus \{0\}$. For given dilation weight $(r_1, \dots, r_n)$, it is easy to verify that the homogeneous 2-norm $\|x\|_\Delta$ is homogeneous of degree 1. By Lemma 4.2 in [14], we have

$$\underline{c}\, \|x\|_\Delta^\tau \leq V(x) \leq \bar{c}\, \|x\|_\Delta^\tau, \tag{2.7}$$

where

$$0 < \underline{c} := \min_{\{z : \|z\|_\Delta = 1\}} V(z) \leq \bar{c} := \max_{\{z : \|z\|_\Delta = 1\}} V(z) < \infty.$$

Let U be a closed set defined by $U = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. If $(x_1, \dots, x_n) \in U$, by using elementary inequalities $(|a|+|b|)^q \leq |a|^q + |b|^q$ when $0 < q \leq 1$ and $(|a|+|b|)^q \leq 2^{q-1}(|a|^q + |b|^q)$ as $q > 1$, we can derive that

$$\sum_{i=1}^n |x_i|^{2/r_i} \leq \sum_{i=1}^n |x_i|^{2/\bar{r}} \leq \begin{cases} \|x\|^{\frac{2}{\bar{r}}}, & \text{if } 0 < \bar{r} \leq 1, \\ 2^{1-\frac{1}{\bar{r}}} \|x\|^{\frac{2}{\bar{r}}}, & \text{if } \bar{r} > 1. \end{cases} \tag{2.8}$$

This, together with (2.7), yields that

$$V(x_1, \dots, x_n) \leq \bar{c}\, \|x\|_\Delta^\tau \leq \bar{c}\Big[2^{\frac{\tau}{2} - \frac{\tau}{2\bar{r}}} \vee 1\Big]\|x\|^{\frac{\tau}{\bar{r}}} := \hat{\bar{c}}\|x\|^{\frac{\tau}{\bar{r}}}, \tag{2.9}$$

where $a \vee b$ means the maximum of a and b. For the case of $(x_1, \dots, x_n) \in U^c$, it follows from (2.8) that

$$\begin{aligned} \|x\|_\Delta &= \Big(\sum_{i=1}^n |x_i|^{2/r_i}\Big)^{1/2} = \Big(\sum_{i=1}^n \Big(\frac{|x_i|}{\|x\|}\Big)^{2/r_i} \|x\|^{2/r_i}\Big)^{1/2} \\ &\leq \|x\|^{\frac{1}{\underline{r}}} \Big(\sum_{i=1}^n \Big(\frac{|x_i|}{\|x\|}\Big)^{2/r_i}\Big)^{1/2} \leq \Big[2^{\frac{1}{2} - \frac{1}{2\bar{r}}} \vee 1\Big]\|x\|^{\frac{1}{\underline{r}}}, \end{aligned}$$

which, together with (2.9), gives the required result.