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Upper/lower bounds of generalized H_2 norms in sampled-data systems with convergence rate analysis and discretization viewpoint

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ABSTRACT

This paper considers linear time-invariant (LTI) sampled-data systems and studies their generalized H_2 norms. They are defined as the induced norms from L_2 to L_∞, in which two types of the L_∞ norm of the output are considered as the temporal supremum magnitude under the spatial ∞-norm and 2-norm. The input/output relation of sampled-data systems is first formulated under their lifting-based treatment. We then develop a method for computing the generalized H_2 norms with operator-theoretic gridding approximation. This method leads to readily computable upper bounds as well as lower bounds of the generalized H_2 norms, whose gaps tend to 0 at the rate of $1/\sqrt{N}$ with the gridding approximation parameter N. An approximately equivalent discretization method of the generalized plant is further provided as a fundamental step to addressing the controller synthesis problem of minimizing the generalized H_2 norms of sampled-data systems. Finally, a numerical example is given to show the effectiveness of the computation method.

1. Introduction

The H_2 norm has been widely used as a performance measure for the disturbance rejection problem. There are two time-domain ideas to define the H_2 norm for linear time-invariant (LTI) sampled-data systems. By this term, we mean that the continuous-time plant is LTI and controlled by an LTI discrete-time controller. Furthermore, our particular interest lies in the intersample behavior of the continuous-time signals about the plant. The first idea [1] considers the L_2 norm of the output for the impulse input occurring at a sampling instant. However, it does not take account of the periodically time-varying nature of LTI sampled-data systems viewed in continuous time. With this nature taken into consideration, the second idea (which admits an equivalent frequency-domain interpretation) [2–4] deals with the root mean square (RMS) of the L_2 norms of all the responses for the impulse inputs occurring at τ in the sampling interval $[0, h)$.

If we confine ourselves to single-output LTI continuous-time and discrete-time systems, on the other hand, we could adopt an alternative time-domain definition of the H_2 norm without considering the impulse input. More precisely, it is well known that the H_2 norm coincides with the induced norm from L_2 to L_∞ [5–7] in such LTI continuous-time systems and that from l_2 to l_∞ [8,9] in such LTI discrete-time systems. Even though this is not the case for the multi-output LTI continuous-time and discrete-time systems, the induced norms have been regarded as their generalized H_2 norms. In such treatment, two different spatial norms (i.e., the vector ∞ and 2 norms) are often dealt with in defining the L_∞ and l_∞ norms of the output for the multi-output cases.

It is well known that the L_2 input considered in the generalized H_2 norms is also relevant to the H_∞ norm of LTI sampled-data systems because the latter is nothing but the induced norm from L_2 to L_2. Even with the same class of inputs, which is practical for dealing with typical disturbances with finite energy, however, taking these two different norms on the output can be quite meaningful depending on different purposes. Use of the L_∞ norm for the output leads to the generalized H_2 norms and is particularly suitable for such situations where having a large instantaneous peak value of the output (in terms of the L_∞ norm) is much more problematic than having a large 'average' value of the output (in terms of the L_2 norm). Thus, considering the generalized H_2 norms properly matches practical control applications such as avoiding robot manipulators from colliding with their surrounding objects and suppressing chemical plants from being overly pressured caused by unknown disturbances with finite energy; the H_∞ and L_1 norms do not match these problems effectively. In this sense, it is worthwhile to study the generalized H_2 norms of LTI sampled-data systems. They were analytically formulated in [10] for the first time by employing the idea of the lifting technique [10–13], but their computation was not discussed there. An approximate

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but asymptotically exact computation method was then provided in [14] by using an idea of fast-sampling rather than lifting.

Recently, the generalized H_2 norms of LTI sampled-data systems were revisited in [15] again with the lifting arguments in such a way that a comparative study of the generalized H_2 norms with the two existing definitions for the H_2 norm of sampled-data systems can also be carried out. The arguments therein revealed for the first time that the generalized H_2 norms coincide with neither of the two existing definitions of the H_2 norm for LTI sampled-data systems. This clearly means that the generalized H_2 norms (i.e., induced norms from L_2 to L_∞ with two alternative underlying spatial norms about the latter) could be interpreted as yet another definition of the H_2 norm of single-output LTI sampled-data systems. However, to compute the generalized H_2 norms with the arguments in [15], one should take the supremum of a suitably constructed function over the sampling interval $[0, h)$. Thus, only an approximate computation approach based on gridding is given in [15] and the arguments in [14] are also subject to the same kind of situation (as well as some more additional assumption on the system). In other words, no upper bounds of the generalized H_2 norms have been derived and their lower bounds are obtained only through gridding in the existing studies.

This paper aims at resolving this issue through an operator-theoretic interpretation of the gridding method [15], and derives computable upper bounds as well as lower bounds of the generalized H_2 norms together with the associated convergence rate. More precisely, such an interpretation leads to a method for computing upper and lower bounds of each of the two generalized H_2 norms for an LTI sampled-data system. Furthermore, it is shown that the gaps between the upper and lower bounds converge to 0 at the rates of $1/\sqrt{N}$, where N is the gridding approximation parameter. This paper also gives an alternative interpretation of the lower bound computation through an approximate discretization process of the generalized plant. To this end, we relate the computation with that of the l_∞/l_2-induced norm of an approximately equivalent LTI discrete-time system constructed with the discretized generalized plant, where the latter computation is readily possible [8,9]. This interpretation through the discretization of the generalized plant gives a crucial basis for the optimal controller synthesis for minimizing the generalized H_2 norms of LTI sampled-data systems [16].

We remark that the gridding approximation has been extended to another approximation approach in [17] but for the case of SISO systems and only for the spatial ∞ norm. The present paper nevertheless opts to confine itself to the simpler gridding approximation at the sacrifice of slight deterioration in the accuracy of approximation while dealing also with the multi-input/multi-output (MIMO) case and the spatial 2 norm; this allows us to circumvent the introduction of the more involved fast-lifting treatment [18] adopted (even for the gridding approach) in [17] and helps us to provide a less complicated perspective for this simpler approach.

The organization of this paper is as follows. Section 2 gives some mathematical notations. In Section 3, the lifting-based results in [15] relevant to the generalized H_2 norms of sampled-data systems are reviewed. The main results of this paper are given in Section 4. A method for computing upper and lower bounds of the generalized H_2 norms together with the associated convergence rate is provided through the operator-theoretic gridding approximation idea. Furthermore, the lower bound computation is related to discretization of the continuous-time generalized plant to open a further direction for the associated optimal controller synthesis. Finally, a numerical example is given in Section 5 to demonstrate the effectiveness of the developed computation method.

2. Mathematical notations

This section gives the mathematical notations used in this paper. The notations $\mathbb{N}$, $\mathbb{R}^\nu_\infty$ and $\mathbb{R}^\nu_2$ denote the set of positive integers, the Banach space of ν-dimensional real vectors equipped with vector ∞-norm (denoted by $|\cdot|_\infty$) and the Hilbert space of ν-dimensional real vectors equipped with the usual inner product and the associated Euclidean norm (denoted by $|\cdot|_2$), respectively. The notation $\mathbb{N}_0$ is further used to imply $\mathbb{N} \cup \{0\}$.

The notations $|\cdot|_{p/2}$ $(p = \infty, 2)$ are used to imply the induced norms of a matrix as a mapping from $\mathbb{R}^{\nu_1}_2$ to $\mathbb{R}^{\nu_2}_p$, i.e.,

$$|T|_{p/2} := \sup_{w \in \mathbb{R}^{\nu_1}_2} \frac{|Tw|_p}{|w|_2} \quad (p = \infty, 2).$$

The notations $\|\cdot\|_{(\infty,p)}$ $(p = \infty, 2)$ are used to mean the $L_\infty[0, h)$ norms under the spatial ∞-norm and 2-norm, respectively, i.e.,

$$\|z(\cdot)\|_{(\infty,\infty)} := \operatorname*{esssup}_{0 \le t < h} |z(t)|_\infty = \operatorname*{esssup}_{0 \le t < h} \max_{1 \le i < \nu} |z_i(t)|,$$

$$\|z(\cdot)\|_{(\infty,2)} := \operatorname*{esssup}_{0 \le t < h} |z(t)|_2 = \operatorname*{esssup}_{0 \le t < h} \left(z^T(t) z(t)\right)^{1/2},$$

or those with h replaced by an integer fraction $h' = h/N$ or ∞, whose distinction will be clear from the context (the same comment applies to the following norm notations used in common for slightly different types of quantities). The notation $\|\cdot\|_{(2,2)}$ is used to mean either the $L_2[0, h)$ norm of a real-vector-valued function, i.e.,

$$\|w(\cdot)\|_{(2,2)} := \left(\int_0^h |w(t)|_2^2 dt\right)^{1/2},$$

or that with h replaced by $h' = h/N$ or ∞.

On the other hand, for an operator $\mathbf{T}$ from $(L_2[0, h))^{\nu_1}$ to $(L_\infty[0, h))^{\nu_2}$, the notations $\|\cdot\|_{(\infty,p)/(2,2)}$ $(p = \infty, 2)$ are used to denote either the induced norms

$$\|\mathbf{T}\|_{(\infty,p)/(2,2)} := \sup_{w \in (L_2[0,h))^{\nu_1}} \frac{\|\mathbf{T}w\|_{(\infty,p)}}{\|w\|_{(2,2)}} \quad (p = \infty, 2),$$

or that with h replaced by ∞. This notation is also applied to the discrete-time case, i.e., the induced norms from l_2 to l_∞ equipped with two spatial (∞ and 2) norms for the l_∞ norm.

The notations $\|\cdot\|_{(\infty,p)/2}$ $(p = \infty, 2)$ are used to imply the induced norms from $\mathbb{R}^{\nu_1}_2$ to $(L_\infty[0, h))^{\nu_2}$, i.e.,

$$\|\mathbf{T}\|_{(\infty,p)/2} := \sup_{x \in \mathbb{R}^{\nu_1}_2} \frac{\|\mathbf{T}x\|_{(\infty,p)}}{|x|_2} \quad (p = \infty, 2)$$

or that with h replaced by h'.

Furthermore, the notation $\|\cdot\|_{2/2}$ is used to imply the induced norm from l_2 or $(L_2[0, h))^{\nu_1}$ to $\mathbb{R}^{\nu_2}_2$, or that with h replaced by h'. Finally, we use the notations $d_{\max}(\cdot)$ and $\lambda_{\max}(\cdot)$ to denote the maximum diagonal entry and maximum eigenvalue of a real symmetric matrix, respectively.

3. Characterization of the generalized H_2 norms in sampled-data systems

This section reviews the results in [15] and gives an explicit characterization for the generalized H_2 norms of LTI sampled-data systems through the lifting treatment [10–13]. A similar result was first derived in [14] without applying the lifting technique.

Let us consider the stable sampled-data system Σ_{SD} shown in Fig. 1, where P represents the continuous-time LTI generalized plant, while Ψ, $\mathcal{H}$ and $\mathcal{S}$ represent the discrete-time LTI controller, the zero-order hold and the ideal sampler, respectively, operating with sampling period h in a synchronous fashion. Solid lines and