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Consensus via multi-population robust mean-field games

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ABSTRACT

In less prescriptive environments where individuals are told 'what to do' but not 'how to do', synchronization can be a byproduct of strategic thinking, prediction, and local interactions. We prove this in the context of multi-population robust mean-field games. The model sheds light on a multi-scale phenomenon involving fast synchronization within the same population and slow inter-cluster oscillation between different populations.

1. Introduction

Synchronization is a natural phenomenon which arises in many applications such as pricing in finance [1,2], opinion dynamics [3], or transient stability of generators [4] etc. Most of the models for synchronization are derived in prescriptive environments in which individuals, the agents, are pre-programmed to adopt specific behaviors, see [5] and references therein.

In this paper we consider a multi-population of dynamic agents as illustrated in Fig. 1.

The dynamics of each agent – henceforth referred to as *microscopic* dynamics – describes the time evolution of its state in the form of a stochastic differential equation. In addition, for each population of agents, we consider the corresponding *phase coherence*, which is a measure of the synchronization of the agents of that population, and the associated dynamics, the latter called *macroscopic* dynamics. Each agent seeks to synchronize its phase to the local average phase obtained via mean-field computation. The model highlights the following aspects: (i) each agent is a rational player equipped with strategic and computation capabilities; (ii) the interaction is local and subject to disturbances; (iii) agents are heterogeneous. Local interaction is determined by geographic proximity between two populations, and is modeled by a network topology where the nodes are the populations and the links establish neighbor relations. The model is a multi-population robust mean-field games within the theory proposed by M.Y. Huang, P. E. Caines and R. Malhamé in [6–8] and independently by Lasry and Lions in [9]. For a survey see [10]. Modeling synchronization as a game is also in [11]. Game theoretic learning is also discussed in [12]. Efficiency loss in equilibria is studied in [13]. Higher level interactions between the subpopulations are analyzed in [14] in the context of auctions. While sharing some of the general concepts already present in the aforementioned references, this paper adds new elements such as local interactions, disturbances and heterogeneity in a unified framework.

Main contribution. This paper shows that synchronization can be obtained in less prescriptive environments as byproduct of strategic thinking, prediction, and local interactions, see Fig. 2. Even if the agents are not pre-programmed to adopt certain strategies, a proper mix of the above three factors will lead to synchronization. To address model misspecification, the game involves the presence of an adversarial disturbance which captures uncertainty in the microscopic dynamics (i.e. some players may be irrational). The resulting game is then a robust mean-field game as the one in [15] and in the same spirit as [16].

The model involves a system of coupled partial differential equations (PDEs). For each population we have one PDE in the form of a Hamilton–Jacobi–Isaacs (HJI) equation, and a second PDE which is the Fokker–Planck–Kolmogorov (FPK) equation describing the diffusion process of the agents' states. We provide a solution of the HJI equation under the assumption that the time evolution of the common state is given. We show that the problem reduces to solving three matrix equations and that in the infinite horizon case the macroscopic dynamics is a typical consensus dynamics.

The analysis of the mean-field game is then extended to the case of second-order dynamics. Even for this case, we prove that the problem of approximating mean-field equilibrium strategies

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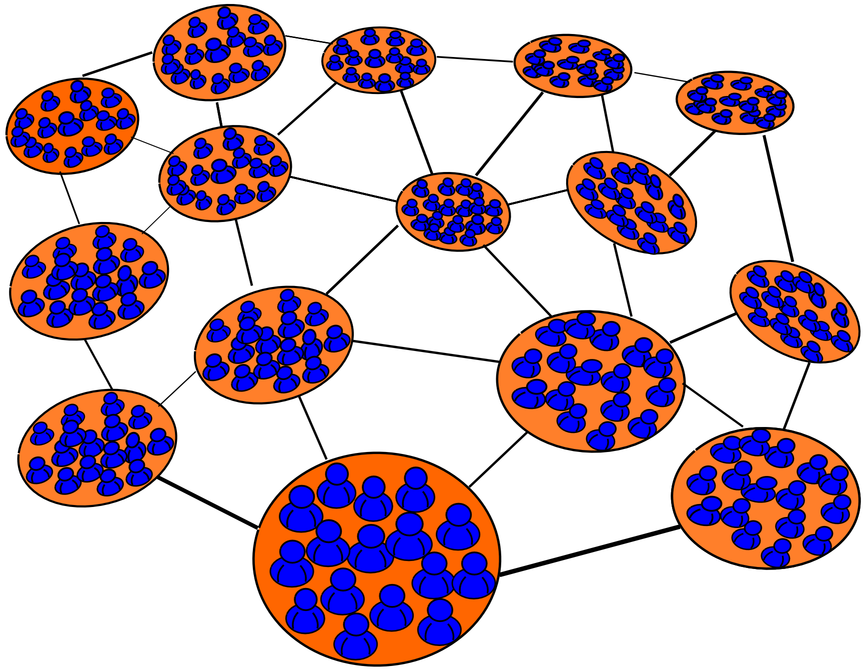

Fig. 1. Multi-population model with local interactions.

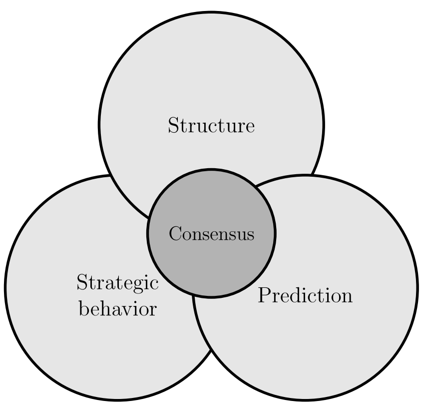

Fig. 2. Synchronization as a result of a proper mix of strategic thinking, prediction, and local interaction in a structured environment.

reduces to solving three matrix equations. By taking the limit for $T \to \infty$ the macroscopic dynamics takes the form of a second-order consensus dynamics. Simulations of simple heuristics show the multi-scale nature of the process involving fast synchronization within the same population and slow inter-cluster oscillation capturing delays due to the geographic sparsity of the populations.

The remainder of the paper is structured as follows. In Section 2 we formulate the problem. In Section 3 we discuss examples. The main results are presented in Sections 4 and 5. Section 6 provides a numerical example. Finally in Section 7 we provide conclusions.

2. Model and problem set-up

Consider p populations of homogeneous agents (players); each player belongs to a population $k \in \{1, \ldots, p\}$ and is characterized by a state $X(t) \in \mathbb{R}$ at time $t \in [0, T]$, where $[0, T]$ is the time horizon window. The control variable is a measurable function of time, $u(\cdot) \in U$, where U is the control set, defined as $t \mapsto \mathbb{R}$ and establishes the rate of variation of an agent's state. A disturbance tries to affect the agents' state in a way that is proportional to his efforts $w(\cdot) \in W$, where W is the control set of the disturbance.

The state dynamics of each player is

$$dX(t) = (u(t) + w(t))dt + \sigma d\mathcal{B}(t), \; t > 0, \tag{1}$$

where $X(0) = x$ for given initial state x, $\sigma > 0$ is a weighting coefficient and $\mathcal{B}(t)$ is the standard Brownian motion process.

For every population $k \in \{1, \ldots, p\}$, consider a probability density function $m_k : \mathbb{R} \times [0, +\infty[\; \to \mathbb{R}$, $(x, t) \mapsto m_k(x, t)$, representing the density of agents of that population in state x at time t, which satisfies $\int_{\mathbb{R}} m_k(x, t)dx = 1$ for every t. Let the *mean state* of population k at time t be $\overline{m}_k(t) := \int_{\mathbb{R}} x m_k(x, t)dx$. From averaging both sides of (1) we get the aggregated dynamics

$$\frac{d}{dt}\overline{m}_k(t) = \overline{u}_k(t) + \overline{w}_k(t),$$

where $\overline{u}_k(t)$ and $\overline{w}_k(t)$ are the mean state-feedback control and disturbance of that population, i.e.,

$$\overline{u}_k(t) := \int_{\mathbb{R}} u(x, t)m_k(x, t)dx, \qquad \overline{w}_k(t) := \int_{\mathbb{R}} w(x, t)m_k(x, t)dx.$$

Let a graph $G = (V, E)$ be given where $V = \{1, \ldots, p\}$ is the set of vertices, one per each population, and $E = V \times V$ is the set of edges. Although most results are easily generalizable to more general graphs, possibly time-varying, for the sake of simplicity we henceforth assume that $G = (V, E)$ is a connected undirected graph, see e.g. [5, Lemma 1]. Denote the set of neighbors of k by $N(k) = \{j \in V \,|\, (k, j) \in E\}$.

The objective of an agent is to adjust his state based on the aggregate k th state. Set

$$\rho_k = \frac{\sum_{j \in N(k)} \overline{m}_j(t)}{|N(k)|}, \tag{2}$$

where $|N(k)|$ denotes the cardinality of the set $N(k)$, namely the number of neighbors of k.

Then, for the agents, consider a running cost $g : \mathbb{R} \times \mathbb{R} \times U \to [0, +\infty[$, and a terminal cost $\Psi : \mathbb{R} \times \mathbb{R} \to [0, +\infty[$, given by:

$$g(x, \rho_k, u) = \frac{1}{2}\left[a(\rho_k - x)^2 + cu^2\right], \tag{3}$$

$$\Psi(\rho_k, x) = \frac{1}{2}S(\rho_k - x)^2. \tag{4}$$