Systems & Control Letters 104 (2017) 1–4

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Explicit solutions for continuous time mean–variance portfolio selection with nonlinear wealth equations

Shaolin Ji [a,b], Xiaomin Shi [a,*]

[a] Zhongtai Securities Institute for Financial Studies, Shandong University, Jinan 250100, China
[b] Institute of Mathematics, Shandong University, Jinan 250100, China

ARTICLE INFO

Article history:
Received 11 June 2016
Received in revised form 22 January 2017
Accepted 20 March 2017

Keywords:
Mean–variance portfolio selection
Nonlinear wealth equation
HJB equation
Viscosity solution

ABSTRACT

This paper concerns the continuous time mean–variance portfolio selection problem with a special nonlinear wealth equation. This nonlinear wealth equation has a nonsmooth coefficient and the dual method developed in Ji (2010) does not work. We invoke the HJB equation of this problem and give an explicit viscosity solution of the HJB equation. Furthermore, via this explicit viscosity solution, we obtain explicitly the efficient portfolio strategy and efficient frontier for this problem. Finally, we show that our nonlinear wealth equation can cover three important cases.

1. Introduction

A mean–variance portfolio selection problem consists in finding the optimal portfolio strategy which minimizes the variance of its terminal wealth while its expected terminal wealth equals a prescribed level. Markowitz [1,2] first studied this problem in the single-period setting. Its multi-period and continuous time counterparts have been studied extensively in the literature; see, e.g. [3–7], and the references therein. Most of the literature on mean–variance portfolio selection focuses on an investor with linear wealth equation. But in some cases, one need to consider nonlinear wealth equations. For example, a large investor's portfolio selection may affect the return of the stock's price which leads to a nonlinear wealth equation. When some taxes must be paid on the gains made on the stocks, we also have to deal with a nonlinear wealth equation.

As for the continuous time mean–variance portfolio selection problem with nonlinear wealth equation, Ji [8] obtained a necessary condition for the optimal terminal wealth when the coefficient of the wealth equation is smooth. [9] studied the continuous time mean–variance portfolio selection problem with higher borrowing rate in which the wealth equation is nonlinear and the coefficient is not smooth. They employed the viscosity solution of the HJB equation to characterize the optimal portfolio strategy.

In this paper, the continuous time mean–variance portfolio selection problem with a special nonlinear wealth equation is studied. This nonlinear wealth equation has a nonsmooth coefficient and can cover the following three important models: the first model is proposed by Jouini and Kallal [10] and El Karoui et al. [11] in which an investor has different expected returns for long and short position of the stock (see Example 4.1); the second one is given in Section 4 of [12] for a large investor (see Example 4.2); the third one is introduced in [13] to study the wealth equation with taxes paid on the gains (see Example 4.3). We invoke the Hamilton–Jacobi–Bellman (HJB for short) equation of this problem and give an explicit viscosity solution of the HJB equation. Furthermore, via this explicit viscosity solution, we obtain explicitly the efficient portfolio strategy and efficient frontier for this problem.

The paper is organized as follows. In Section 2, we formulate the problem. Our main results are given in Section 3. In Section 4, we show that our wealth equation (2.1) can cover three important cases.

2. Formulation of the problem

Let W be a standard 1-dimensional Brownian motion defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, where $\{\mathcal{F}_t\}_{t\geq 0}$ denotes the natural filtration associated with the 1-dimensional Brownian motion W and augmented. We denote by $L^2(0,T)$ the space of all $\mathcal{F}_t$−progressively measurable $\mathbb{R}$ valued processes x such that $E\int_0^T x_t^2 dt < \infty$.

We consider a financial market consisting of a riskless asset (the money market instrument or bond) whose price is S^0 and one risky security (the stock) whose price is S^1. An investor can decide at time $t \in [0,T]$ what amount π_t of his wealth X_t to invest in the stock. Of course, his decisions can only depend on the current information $\mathcal{F}_t$, that is, the portfolio π is $\mathcal{F}_t$-adapted.

* Corresponding author.
E-mail addresses: jsl@sdu.edu.cn (S. Ji), shixm@mail.sdu.edu.cn, shixiaominhi@163.com (X. Shi).

http://dx.doi.org/10.1016/j.sysconle.2017.03.006

For given deterministic continuous functions $r_t, \underline{\theta}_t, \bar{\theta}_t, \sigma_t$ on $[0, T]$, consider the following nonlinear wealth equation:

$$\begin{cases} dX_t = (r_t X_t + \pi_t^+ \sigma_t \underline{\theta}_t - \pi_t^- \sigma_t \bar{\theta}_t)dt + \pi_t \sigma_t dW_t, \\ X_0 = x_0, \ t \in [0, T] \end{cases} \tag{2.1}$$

where the functions $x^+ := \begin{cases} x, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0, \end{cases}$ and $x^- := \begin{cases} -x, & \text{if } x \leq 0; \\ 0, & \text{if } x > 0. \end{cases}$

In general, r is the risk-free interest rate and σ is the standard deviation of the stock's returns. $\underline{\theta}$ and $\bar{\theta}$ are the stock's risk premiums for the long position and the short position respectively. In many situations, $\underline{\theta}$ and $\bar{\theta}$ could be different directly or indirectly. In Section 4, we will give three financial models in which this kind of wealth equation emerges and the concrete meaning of $\underline{\theta}$ and $\bar{\theta}$ in each example.

We assume:

Assumption 2.1. $0 \leq \underline{\theta}_t \leq \bar{\theta}_t$, a.s. on $[0, T]$, $\sigma_t \neq 0$, a.s. on $[0, T]$.

Remark 2.2. When $\underline{\theta}_t = \bar{\theta}_t$, a.s. on $[0, T]$, the wealth equation (2.1) reduces to the classical self-financing linear wealth equation.

For a given expectation level K, consider the following continuous time mean–variance portfolio selection problem:

$$\text{Minimize } VarX_T = E(X_T - K)^2,$$
$$s.t. \begin{cases} EX_T = K, \\ \pi \in L^2(0, T), \\ (X, \pi) \text{ satisfies Eq. (2.1).} \end{cases} \tag{2.2}$$

Throughout the paper, we assume that $K \geq x_0 e^{\int_0^T r_s ds}$. Notice that the investor will get $x_0 e^{\int_0^T r_s ds}$ if all his/her initial wealth x_0 were invested in the bond, that is $\pi_t = 0$, $t \in [0, T]$, a.s. So problem (2.2) under $K < x_0 e^{\int_0^T r_s ds}$ is meaningless for rational investors.

The optimal strategy π^* is called an efficient strategy. Denote the optimal terminal value by X_T^*. Then, $(VarX_T^*, K)$ is called an efficient point. The set of all efficient points $\{(VarX_T^*, K) \mid K \in [x_0 e^{\int_0^T r_s ds}, +\infty)\}$ is called the efficient frontier.

Definition 2.3. A portfolio π is said to be admissible if $\pi \in L^2(0, T)$ and (X, π) satisfies Eq. (2.1).

Denote by $\mathcal{A}(x_0; 0, T)$ the set of portfolio π admissible for the initial investment x_0. For simplicity, we set $\mathcal{A}(x_0) := \mathcal{A}(x_0; 0, T)$.

3. Main results

To deal with the constraint $EX_T = K$, we introduce a Lagrange multiplier $-2\lambda \in \mathbb{R}$ and get the following auxiliary optimal stochastic control problem:

$$\text{Minimize } E(X_T - K)^2 - 2\lambda(EX_T - K)$$
$$= E(X_T - d)^2 - (d - K)^2 =: \hat{J}(\pi, d),$$
$$s.t. \begin{cases} \pi \in L^2(0, T), \\ (X, \pi) \text{ satisfies Eq.(2.1),} \end{cases} \tag{3.1}$$

where $d := K + \lambda$.

Remark 3.1. The link between problem (2.2) and (3.1) is provided by the Lagrange duality theorem (see Luenberger [14])

$$\min_{\pi \in \mathcal{A}(x_0), EX_T = K} VarX_T = \max_{d \in \mathbb{R}} \min_{\pi \in \mathcal{A}(x_0)} \hat{J}(\pi, d).$$

So the optimal problem (2.2) can be divided into two steps. The first step is to solve

$$\text{Minimize } E(X_T - d)^2, \ s.t. \ \pi \in \mathcal{A}(x_0), \tag{3.2}$$

for any fixed $d \in \mathbb{R}$. The second step is to find the Lagrange multiple which attains

$$\max_{d \in \mathbb{R}} \min_{\pi \in \mathcal{A}(x_0)} \hat{J}(\pi, d).$$

To solve the first step, we introduce the stochastic control problem

$$v(t, x; d) := \inf_{\pi \in \mathcal{A}(x; t, T)} E(X_T - d)^2, \ (t, x) \in [0, T] \times \mathbb{R} \tag{3.3}$$

on $[t, T]$, subject to

$$\begin{cases} dX_s = (r_s X_s + \pi_s^+ \sigma_s \underline{\theta}_s - \pi_s^- \sigma_s \bar{\theta}_s)ds + \pi_s \sigma_s dW_s, \\ X_t = x. \end{cases} \tag{3.4}$$

The value function $v(t, x; d)$ is a viscosity solution of the following HJB equation (refer to [15]):

$$\begin{cases} \dfrac{\partial v}{\partial t}(t, x; d) + \inf_{\pi \in \mathbb{R}} \Big[\dfrac{\partial v}{\partial x}(t, x; d)(r_t x + \pi^+ \sigma_t \underline{\theta}_t - \pi^- \sigma_t \bar{\theta}_t) \\ \quad + \dfrac{1}{2} \dfrac{\partial^2 v}{\partial x^2}(t, x; d) \sigma_t^2 \pi^2 \Big] = 0, \\ v(T, x; d) = (x - d)^2. \end{cases}$$

Note that the functions x^+ and x^- are nonsmooth. Then Eq. (3.5) does not have a smooth solution. In the 1980s, Crandall and Lions [16] introduced the notion of viscosity solutions for first order equations. Then Lions [17–19] studied viscosity solutions in infinite-dimensional spaces. We mainly refer to Crandall, Ishii and Lions [20] for a seminal reference on this topic.

In the following theorem, we construct a viscosity solution of Eq. (3.5).

Theorem 3.2. *Under Assumption 2.1, a viscosity solution of the above HJB equation (3.5) is given by*

$$v(t, x; d) = \begin{cases} e^{-\int_t^T \underline{\theta}_s^2 ds}(x e^{\int_t^T r_s ds} - d)^2, & \text{if } x \leq d e^{-\int_t^T r_s ds}; \\ e^{-\int_t^T \bar{\theta}_s^2 ds}(x e^{\int_t^T r_s ds} - d)^2, & \text{if } x > d e^{-\int_t^T r_s ds}, \end{cases} \tag{3.6}$$

and the associated optimal feedback control is given by

$$\pi^*(t, x) = \begin{cases} -\dfrac{\underline{\theta}_t}{\sigma_t}(x - d e^{-\int_t^T r_s ds}), & \text{if } x \leq d e^{-\int_t^T r_s ds}; \\ -\dfrac{\bar{\theta}_t}{\sigma_t}(x - d e^{-\int_t^T r_s ds}), & \text{if } x > d e^{-\int_t^T r_s ds}. \end{cases} \tag{3.7}$$

Proof. Firstly we verify that the function $v(t, x; d)$ defined in (3.6) is a viscosity solution of HJB equation (3.5). We divide $[0, T] \times \mathbb{R}$ into three disjoint regions

$$\Gamma_1 := \{(t, x) \in [0, T] \times \mathbb{R} | x < d e^{-\int_t^T r_s ds}\};$$
$$\Gamma_2 := \{(t, x) \in [0, T] \times \mathbb{R} | x > d e^{-\int_t^T r_s ds}\};$$
$$\Gamma_3 := \{(t, x) \in [0, T] \times \mathbb{R} | x = d e^{-\int_t^T r_s ds}\}.$$

Notice that the function $v(t, x; d)$ defined in (3.6) is convex, and $v(t, x; d) \in C^{1,2}(\Gamma_1 \cup \Gamma_2)$.

Let $\phi \in C^\infty([0, T] \times \mathbb{R})$, and $(t, x) \in [0, T] \times \mathbb{R}$ is a minimum point of $\phi - v$.

If $(t, x) \in \Gamma_1$, the first- and second-optimality conditions imply

$$\frac{\partial \phi}{\partial t}(t, x) = \frac{\partial v}{\partial t}(t, x)$$
$$= \underline{\theta}_t^2 e^{-\int_t^T \underline{\theta}_s^2 ds}(x e^{\int_t^T r_s ds} - d)^2 - 2 x r_t e^{\int_t^T (r_s - \underline{\theta}_s^2) ds}(x e^{\int_t^T r_s ds} - d); \tag{3.8}$$

$$\frac{\partial \phi}{\partial x}(t, x) = \frac{\partial v}{\partial x}(t, x) = 2 e^{\int_t^T (r_s - \underline{\theta}_s^2) ds}(x e^{\int_t^T r_s ds} - d); \tag{3.9}$$

$$\frac{\partial^2 \phi}{\partial x^2}(t, x) \geq \frac{\partial^2 v}{\partial x^2}(t, x) = 2 e^{\int_t^T (2 r_s - \underline{\theta}_s^2) ds}. \tag{3.10}$$