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Nonlinear sliding mode control design: An LMI approach

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ABSTRACT

This paper presents a novel nonlinear sliding mode control methodology for systems with both matched and unmatched perturbations (including parametric uncertainties). Instead of traditional approaches where uncertainties and nonlinearities are coped with via linear nominal models and linear sliding surfaces, the proposed approach incorporates exact convex expressions to represent both the nonlinear surface and the system, thus allowing a significant chattering reduction. Moreover, thanks to the convex form of the nonlinear nominal model, when combined with the direct Lyapunov method, it leads to linear matrix inequalities, which are efficiently solved via convex optimization techniques. Illustrative examples are provided.

1. Introduction

The main characteristics of sliding mode control (SMC) are insensitivity to matched disturbances and finite-time convergence to the sliding surface [1]; these benefits have sustainedly allured researchers in control systems for several decades, leading to increasingly complex control laws which intend to minimize the side effects of high frequency signals, this is to say, chattering [2–4] and magnitude of the control law. Ordinarily, a system is decomposed into a linear nominal system plus affine terms where nonlinearities and uncertainties (both matched and unmatched) are grouped; the sliding surface is chosen as a linear combination of the states since this eases the development of the basic theory [5].

Rejection of unmatched uncertainties and perturbations is an important task in standard SMC [6]. Some approaches based on backstepping ensure only exact tracking of the output [7,8]; others only minimization of their influence [9,10]. Nevertheless, a more realistic and less conservative approach might be to deal with a *nonlinear nominal system*, because a linear one subsumes a *family* of models into a single one, thus lacking specificity; this has been already pointed out in [1,11], where a system is shown to converge more rapidly to a nonlinear sliding surface than to a linear one, but this example is far from being systematic. Additional advantages of keeping a nonlinear nominal system can also be foreseen: if some of the affine terms usually disregarded as matched and unmatched perturbations and parametric uncertainties are kept into the nominal system, it might happen their unmatched quality will disappear, thus diminishing the size of the control signal while preserving insensitivity to matched uncertainties. Since the chattering effect is directly connected with the size of the uncertainties, the proposed ideas are fairly justified.

Moreover, how to reach the systematic character of the linear nominal-based methodologies if a nonlinear one is employed instead? The answer hereby proposed is based on exact convex representations of nonlinear terms, a technique well known in the linear parameter varying (LPV) and quasi-LPV literature [12–14] and successfully extended for convex sums of linear [15,16] and polynomial models [17]. These representations are *not approximations*. They have led to full developed and still active Lyapunov-based nonlinear methodologies with the additional advantage of expressing their conditions in the form of linear matrix inequalities (LMIs), which belong to the class of convex optimization problems [18,19] that can be solved with commercially available software [20,21].

As shown in this paper, the use of convex structures for SMC design allows working with nonlinear expressions by mimicking the linear case; matched and unmatched uncertainties as well as parametric ones can be exactly dealt with instead of discarded or approximated. This advantage reduces the chattering effect, since the size of the control gain is diminished. Moreover, this approach inherits the LMI quality of solutions. This is not the first time LMI conditions have been proposed for SMC design: we found them in several works [22–26], but to the best of our knowledge, none of these works is concerned with nonlinear nominal systems and nonlinear sliding surfaces.

The paper is organized as follows: Section 2 defines the sort of nonlinear systems this work is concerned about and explains how

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they can be equivalently rewritten as a convex sum of linear models, which leads to a motivational example as well as a problem statement; Section 3 develops the main contributions of this work: nonlinear sliding surface design as well as SMC methodologies based on nonlinear nominal systems, both certain and uncertain, with matched and unmatched disturbances; Section 4 provides illustrative examples to point out the effectiveness of the proposed results; the final part, Section 5, draws some conclusions and discusses future work.

2. Preliminaries

Consider the nonlinear affine-in-control system

$$\dot{\chi}(t) = f(\chi) + g(\chi)\left(u(t) + \tilde{\zeta}(t,\chi)\right) \tag{1}$$

where $\chi(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $\tilde{\zeta} : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^m$ are matched uncertainties, $f(\cdot)$ and $g(\cdot)$ are smooth nonlinear vector fields of adequate size.

In [27] a diffeomorphism $T(\cdot)$ was proposed such that the system (1) can be transformed into a regular form:

$$\begin{aligned}\dot{\eta} &= a_{11}(\eta,\xi)\eta + a_{12}(\eta,\xi)\xi \\ \dot{\xi} &= a_{21}(\eta,\xi)\eta + a_{22}(\eta,\xi)\xi + b_2(\eta,\xi)\left(u + \zeta(t,\eta,\xi)\right)\end{aligned} \tag{2}$$

with $\eta \in \mathbb{R}^{n-m}$, $\xi \in \mathbb{R}^m$, matrix functions $a_{jk}(\cdot,\cdot)$, $j,k \in \{1,2\}$, of adequate size, and $b_2(\cdot,\cdot) \in \mathbb{R}^{m\times m}$ being nonsingular for all (η,ξ) in a neighbourhood of the origin.

Lemma 1. *Smooth bounded nonlinear expressions $z(\cdot) \in \left[\underline{z}, \overline{z}\right]$ can always be written as convex sums of their bounds.*

Proof. Define $w_0(z) = (\overline{z} - z)/(\overline{z} - \underline{z})$ and $w_1(z) = 1 - w_0(z)$; then $z = \underline{z}w_0(z) + \overline{z}w_1(z)$. The latter is a convex sum within the interval under consideration since $0 \le w_0(z) \le 1$, $0 \le w_1(z) \le 1$, $w_0(z) + w_1(z) = 1$ (convex sum property). □

For instance, if $z(x) = x^2$ is considered for $x \in [-1, 2]$, then $z(x) = w_0(z)\underline{z} + w_1(z)\overline{z}$ with $w_0(z) = \left(\overline{z} - z\right)/\left(\overline{z} - \underline{z}\right)$, $w_1(z) = 1 - w_0(z) = \left(z - \underline{z}\right)/\left(\overline{z} - \underline{z}\right)$, $\overline{z} = 4$, $\underline{z} = 0$. Note that this algebraic rewriting is not an approximation and preserves convexity within the interval under consideration.

Convex sums such as the one just described can be grouped at the leftmost side of an expression. For instance, if $w_{(0,1)} + w_{(1,1)} = 1$ and $w_{(0,2)} + w_{(1,2)} = 1$, then, for given $X_i, Y_i \in \mathbb{R}^n$, $i \in \{0,1\}$:

$$\sum_{i=0}^{1} w_{(i,1)}X_i + \sum_{i=0}^{1} w_{(i,2)}Y_i = \sum_{i=0}^{1}\sum_{j=0}^{1} w_{(i,1)}w_{(j,2)}(X_i + Y_j) = \sum_{i=1}^{4} h_i Z_i,$$

where $h_1 = w_{(0,1)}w_{(0,2)}$, $h_2 = w_{(0,1)}w_{(1,2)}$, $h_3 = w_{(1,1)}w_{(0,2)}$, $h_4 = w_{(1,1)}w_{(1,2)}$, $Z_1 = X_0 + Y_0$, $Z_2 = X_0 + Y_1$, $Z_3 = X_1 + Y_0$, $Z_4 = X_1 + Y_1$.

The previous considerations allow rewriting the dynamical model (2) as a convex sum of linear ones, where nonlinearities are captured in functions that hold the convex sum property. Such forms are *exact representations of the nonlinear model in a compact set of the state space* and are quite common in the quasi-LPV literature [14,15]. The methodology just described is also known as the *sector nonlinearity approach*.

Applying such methodology to the regular form in (2) requires performing the following steps:

1. Identify the p non-constant bounded terms $z_j(\eta,\xi) \in \left[\underline{z}_j, \overline{z}_j\right]$, $j \in \{1,2,\ldots,p\}$, in expressions $a_{jk}(\eta,\xi)$, $j,k \in \{1,2\}$, $b_2(\eta,\xi)$.
2. Construct p pairs of functions $w_{(0,j)}$, $w_{(1,j)} = 1 - w_{(0,j)}$, $j \in \{1,2,\ldots,p\}$ such that $w_{(0,j)}(z_j) = (\overline{z}_j - z_j)/(\overline{z}_j - \underline{z}_j)$.
3. Provided that $z(\eta,\xi) = \begin{bmatrix} z_1 & z_2 & \cdots & z_p\end{bmatrix}^T$, construct $r = 2^p$ functions $h_i(z)$, $i \in \{1,2,\ldots,r\}$ such that
$$h_i(z) = h_{1+i_1+i_2\times 2+\cdots+i_p\times 2^{p-1}}(z) = \prod_{j=1}^{p} w_{(i_j,j)}(z_j).$$
4. Construct matrices A_i^{jk}, B_i^2, $i \in \{1,2,\ldots,r\}$, $j,k \in \{1,2\}$, such that $A_i^{jk} = a_{jk}(\eta,\xi)\Big|_{h_i=1}$, $B_i^2 = b_2(\eta,\xi)\Big|_{h_i=1}$.

Once these steps are completed, system (2) can be *equivalently* written as a *regular convex model*:

$$\begin{aligned}\dot{\eta} &= A_h^{11}\eta + A_h^{12}\xi \\ \dot{\xi} &= A_h^{21}\eta + A_h^{22}\xi + B_h^2\left(u + \zeta(t,\eta,\xi)\right)\end{aligned} \tag{3}$$

where $A_h^{jk} = \sum_{i=1}^{r} h_i(z)A_i^{jk}$, $j,k \in \{1,2\}$, $B_h^2 = \sum_{i=1}^{r} h_i(z)B_i^2$. Clearly, B_h^2 inherits the invertibility properties of $b_2(\eta,\xi)$.

We can compactly write (3) as follows:

$$\dot{x} = A_h x + B_h\left(u + \zeta(t,x)\right), \tag{4}$$

$$\text{with} \quad x = \begin{bmatrix}\eta \\ \xi\end{bmatrix}, \quad A_h = \begin{bmatrix} A_h^{11} & A_h^{12} \\ A_h^{21} & A_h^{22}\end{bmatrix}, \quad B_h = \begin{bmatrix} 0 \\ B_h^2 \end{bmatrix}.$$

Remark 1. Keep in mind that any expression with h as a subscript is in general a nonlinear one, i.e., though the structure in (3) and (4) reminds that of a "linear" one, they actually preserve all the information (including nonlinearities) of their original form (2).

Motivation: Ordinarily, sliding mode control methodologies consider a linear nominal system of the form $\dot{x} = Ax + Bu$, grouping perturbations and uncertainties in a term which is usually split into matched/unmatched parts: once sliding mode occurs, the system is made insensitive to the first sort of perturbations; H_∞ is usually employed to tackle the second class of perturbations. If nonlinear systems are successfully controlled stacking nonlinearities as uncertainties as proved in many academic as well as practical examples, what is the motivation behind nonlinear convex representations such as (4)? The answer is illustrated with the following example:

$$\begin{aligned}\dot{\eta}_1 &= -\eta_1 + \eta_1\eta_2 \\ \dot{\eta}_2 &= \eta_2 + \xi \\ \dot{\xi} &= \eta_1^2 + u(t) + \zeta(t,\eta_1,\eta_2,\xi),\end{aligned} \tag{5}$$

where $\zeta(t,\eta_1,\eta_2,\xi)$ is an unknown locally Lipschitz function. As mentioned above, traditional sliding mode control methodologies split a nonlinear system into a linear nominal one plus nonlinear/uncertain/disturbance terms; i.e.:

$$\begin{aligned}\begin{bmatrix}\dot{\eta}_1 \\ \dot{\eta}_2\end{bmatrix} &= \begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}\eta_1 \\ \eta_2\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}\xi + \begin{bmatrix}\eta_1\eta_2 \\ 0\end{bmatrix} \\ \dot{\xi} &= u + \zeta(t,\eta_1,\eta_2,\xi) + \eta_1^2,\end{aligned} \tag{6}$$

which clearly includes matched as well as unmatched uncertainties. The sliding surface is then defined as $s = s_1\eta_1 + s_2\eta_2 + \xi$, which in turn defines the nonlinear part of the control law $u_n = -\rho\left(t,\eta_1,\eta_2,\xi\right)\operatorname{sgn}(s)$, with $\rho(t,\eta_1,\eta_2,\xi)$ begin greater or equal to a function of bounds on $\zeta(t,\eta_1,\eta_2,\xi) + \eta_1^2$ (matched) as well as $\eta_1\eta_2$ (unmatched) [5].

Now, consider the following rewriting of (5):

$$\begin{bmatrix}\dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\xi}\end{bmatrix} = \left[\begin{array}{cc|c} -1 & \eta_1 & 0 \\ 0 & 1 & 1 \\ \hline 3\eta_1 & 0 & 0\end{array}\right]\begin{bmatrix}\eta_1 \\ \eta_2 \\ \xi\end{bmatrix} + \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix}\left(u(t) + \zeta(t,\eta_1,\eta_2,\xi)\right). \tag{7}$$