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New results on the Stackelberg–Nash exact control of linear parabolic equations

F.D. Araruna [a], E. Fernández-Cara [b], S. Guerrero [c], M.C. Santos [d,*]

[a] Dpto. de Matemática, Universidade Federal da Paraíba, 58051-900, João Pessoa - PB, Brazil
[b] Dpto. EDAN and IMUS, University of Sevilla, Aptdo. 1160, 41080 Sevilla, Spain
[c] Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 75252 Paris Cédex 05, France
[d] Dpto. de Matemática, Universidade Federal da Pernambuco, 50670-901, Recife - PE, Brazil

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Dedicated to Professor L.A. Medeiros on the occasion of his 90th birthday.

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ABSTRACT

This paper is concerned with Stackelberg–Nash strategies to control parabolic equations. We have one control, the *leader*, that is responsible for a null controllability property; additionally, we have a couple of controls, called the *followers*, that provides a *Nash equilibrium* for two cost functionals. This is a classical situation in many fields of science and, in mathematics, leads to a lot of interesting questions and open problems and possesses many applications. In the main result, we prove the existence of a leader such that the corresponding controlled system is driven to zero. This way, we improve some questions that were left open in previous works.

1. Introduction

There are plenty of situations where several controls are required in order to drive a system to one or more objectives. Usually, if we assign different roles to the controls, we speak of *hierarchic control*. In the case of a system governed by a PDE, this concept was introduced by J.-L. Lions (see [1,2], where some techniques are presented). These works motivated the study of the subject and a lot of other results appeared; see for instance [3–7].

All these previous works combine the multicriteria optimization concepts and arguments and approximate controllability. In the context of null controllability, few is known; see [8] for some first results.

In this paper, we solve a question that was left open in [8]. The solution requires some careful computations based on new Carleman estimates. Let us be more precise.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain whose boundary Γ is regular enough. Let $T > 0$ be given and define $Q := \Omega \times (0,T)$, with lateral boundary $\Sigma := \partial\Omega \times (0,T)$. In the sequel, we will denote by C a generic positive constant which may differ from line to line. Sometimes, we will write $C(\Omega)$, $C(\Omega, T)$, etc. to indicate the data on which C depends. The usual norm and scalar product in $L^2(\Omega)$ will be respectively denoted by $\|\cdot\|$ and $(\cdot\,,\cdot)$.

Let us consider the linear system

$$\begin{cases} y_t - \Delta y + a(x,t)y = f 1_{\mathcal{O}} + v^1 1_{\mathcal{O}_1} + v^2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(\cdot,0) = y^0 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where $y = y(x,t)$ is the state, $a \in L^\infty(Q)$ and $y^0 = y^0(x)$ is prescribed. In (1), the set $\mathcal{O} \subset \Omega$ is the *main control domain* and $\mathcal{O}_1, \mathcal{O}_2 \subset \Omega$ are the *secondary control domains* (all of them are supposed to be small); $1_{\mathcal{O}}$, $1_{\mathcal{O}_1}$ and $1_{\mathcal{O}_2}$ are the characteristic functions of $\mathcal{O}$, $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively; the controls are the *leader* $f = f(x,t)$ and the *followers* $v^1 = v^1(x,t)$ and $v^2 = v^2(x,t)$.

Let $\mathcal{O}_{1,d}, \mathcal{O}_{2,d} \subset \Omega$ be open sets, representing observation domains for the followers. We will consider the (secondary) functionals

$$J_i(f; v^1, v^2) := \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} \left|y - y_{i,d}\right|^2 dx\,dt + \frac{\mu}{2} \iint_{\mathcal{O}_i\times(0,T)} |v^i|^2\,dx\,dt, \quad i = 1, 2,$$

and the main functional

$$J(f) := \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |f|^2 dx\,dt,$$

* Corresponding author.
E-mail addresses: fagner@mat.ufpb.br (F.D. Araruna), cara@us.es (E. Fernández-Cara), guerrero@ann.jussieu.fr (S. Guerrero), mauricio@dmat.ufpe.br (M.C. Santos).

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where the $\alpha_i > 0$ and $\mu > 0$ are constants and the $y_{i,d} = y_{i,d}(x, t)$ are given functions.

The structure of the control process can be described as follows:

1. For each leader f, the followers v^1 and v^2 intend to be a *Nash equilibrium* for the costs J_i $(i = 1, 2)$. In other words, once f has been fixed, we look for a couple (v^1, v^2) with $v^i \in L^2(\mathcal{O}_i \times (0, T))$ such that
$$\begin{aligned} J_1(f; v^1, v^2) &= \min_{\hat{v}^1} J_1(f; \hat{v}^1, v^2), \\ J_2\left(f; v^1, v^2\right) &= \min_{\hat{v}^2} J_2(f; v^1, \hat{v}^2). \end{aligned} \tag{2}$$
Note that, if the functionals J_i $(i = 1, 2)$ are C^1 and convex, then (v^1, v^2) is a Nash equilibrium if and only if
$$J_1'(f; v^1, v^2)(\hat{v}^1, 0) = 0, \\ \forall \hat{v}^1 \in L^2\left(\mathcal{O}_1 \times (0, T)\right), \quad v^i \in L^2(\mathcal{O}_i \times (0, T))$$
and
$$J_2'(f; v^1, v^2)(0, \hat{v}^2) = 0, \\ \forall \hat{v}^2 \in L^2\left(\mathcal{O}_2 \times (0, T)\right), \quad v^i \in L^2(\mathcal{O}_i \times (0, T)).$$
(In fact, this is also true if J_i is C^1 and convex in the ith variable.)
2. Let us fix an uncontrolled trajectory of (1), that is, a sufficiently regular solution to the system
$$\begin{cases} \bar{y}_t - \Delta \bar{y} + a(x, t)\bar{y} = 0 & \text{in} \quad Q, \\ \bar{y} = 0 & \text{on} \quad \Sigma, \\ \bar{y}(\cdot, 0) = \bar{y}^0 & \text{in} \quad \Omega. \end{cases} \tag{3}$$
Once the Nash equilibrium has been identified and fixed for each f, we look for an optimal control $\hat{f} \in L^2(\mathcal{O} \times (0, T))$ such that
$$J(\hat{f}) = \min_f J(f),$$
subject to the exact controllability restriction
$$y(\cdot, T) = \bar{y}(\cdot, T) \quad \text{in} \quad \Omega. \tag{4}$$

In [8] it is proved that, if μ is large enough, for every $f \in L^2(\mathcal{O} \times (0, T))$ there exists a unique Nash equilibrium (v^1, v^2) for (J_1, J_2), given by
$$v_i = -\frac{1}{\mu}\phi^i 1_{\mathcal{O}_i}, \quad i = 1, 2,$$
where (y, ϕ^1, ϕ^2) is the unique solution to the optimality system
$$\begin{cases} y_t - \Delta y + a(x, t)y = f\, 1_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu}\phi^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ -\phi^i_t - \Delta \phi^i + a(x, t)\phi^i = \alpha_i(y - y_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ y = 0, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ y(\cdot, 0) = y^0, \quad \phi^i(\cdot, T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{5}$$

The main result of this paper concerns the exact controllability to the trajectories of (1)–(2). It is the following:

Theorem 1. *Suppose that*
$$\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset, \quad i = 1, 2. \tag{6}$$
Also, assume that one of the following two conditions holds:
$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} \tag{7}$$
or
$$\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \tag{8}$$

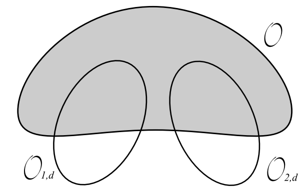

Fig. 1. $\mathcal{O}_{1,d}$ and $\mathcal{O}_{2,d}$ are disjoint.

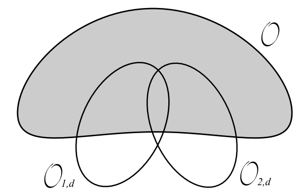

Fig. 2. $\mathcal{O}_{1,d}$ and $\mathcal{O}_{2,d}$ are not disjoint and their intersection cuts $\mathcal{O}$.

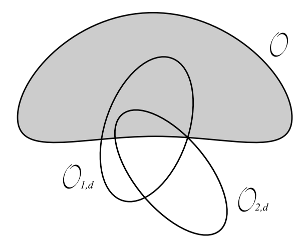

Fig. 3. $\mathcal{O}_{1,d}$ and $\mathcal{O}_{2,d}$ are not disjoint, their intersection cuts $\mathcal{O}$ and their individual intersections with $\mathcal{O}$ are ordered.

Then, there exists $\mu_0 > 0$, only depending on Ω, $\mathcal{O}$, T, $\mathcal{O}_i$, $\mathcal{O}_{i,d}$, α_i and $\|a\|_{L^\infty(Q)}$ and a positive function $\hat{\rho} = \hat{\rho}(t)$ blowing up at $t = T$ such that, if $\mu \geq \mu_0$, the $y_{i,d}$ are such that
$$\iint_{\mathcal{O}_{i,d} \times (0,T)} \hat{\rho}^2 |\bar{y} - y_{i,d}|^2 \, dx \, dt < +\infty, \quad i = 1, 2$$
and $\bar{y}$ is the unique solution to (3) associated to the initial state $\bar{y}^0 \in L^2(\Omega)$, there exist controls $f \in L^2(\mathcal{O} \times (0, T))$ and associated Nash equilibria (v^1, v^2) such that the corresponding solutions to (1) satisfy (4).

Remark 2. It is worth mentioning that, in [8], the authors have proved this result in the particular case in which (6) and (7) are satisfied. Figs. 1–3 illustrate some situations where this fails and (6) and (8) hold simultaneously. □

Note that, if we introduce the new variable $z = y - \bar{y}$, (5) can be rewritten in the form
$$\begin{cases} z_t - \Delta z + a(x, t)z = f\, 1_{\mathcal{O}} - \displaystyle\sum_{i=1}^{2} \frac{1}{\mu}\phi^i 1_{\mathcal{O}_i} & \text{in} \quad Q, \\ -\phi^i_t - \Delta \phi^i + a(x, t)\phi^i = \alpha_i(z - z_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ z = 0, \quad \phi^i = 0 & \text{on} \quad \Sigma, \\ z(\cdot, 0) = z^0, \quad \phi^i(\cdot, T) = 0 & \text{in} \quad \Omega, \end{cases} \tag{9}$$
where $z_{i,d} = y_{i,d} - \bar{y}$ and $z^0 = y^0 - \bar{y}^0$ and (4) is equivalent to the null controllability property for z, that is,
$$z(\cdot, T) = 0 \quad \text{in} \quad \Omega. \tag{10}$$

The proof of Theorem 1 relies on some duality arguments which reduce the null controllability property of a linear system to an observability inequality for the solutions to the associated adjoint