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Sparsity-promoting optimal control of systems with symmetries, consensus and synchronization networks[☆]

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ABSTRACT

Optimal control problems in systems with symmetries and consensus/synchronization networks are characterized by structural constraints that arise either from the underlying group structure or the lack of absolute measurements for part of the state vector. Our objective is to design controller structures and resulting control strategies that utilize limited information exchange between subsystems in large-scale networks. To obtain controllers with low communication requirements, we seek solutions to regularized versions of the $\mathcal{H}_2$ optimal control problem. Non-smooth regularization terms are introduced to tradeoff network performance with sparsity of the feedback-gain matrix. In contrast to earlier results, our framework allows the state-space representations that are used to quantify the system's performance and sparsity of the static output-feedback controller to be expressed in different sets of coordinates. We show how alternating direction method of multipliers can be leveraged to exploit the underlying structure and compute sparsity-promoting controllers. In particular, for spatially-invariant systems, the computational complexity of our algorithm scales linearly with the number of subsystems. We also identify a class of optimal control problems that can be cast as semidefinite programs and provide an example to illustrate our developments.

1. Introduction

In large networks of dynamical systems centralized information processing may impose prohibitively expensive communication and computation burden [1,2]. This motivates the development of theory and techniques for designing distributed controller architectures that lead to favorable performance of large-scale networks. Recently, regularized versions of standard optimal control problems were introduced as a means for achieving this goal [3–6]. For example, in consensus and synchronization networks, it is of interest to achieve desired objective using relative information exchange between limited subset of nodes [7–18].

The objective of this paper is to design controllers that provide a desired tradeoff between the network performance and the sparsity of the static output-feedback controller. This is accomplished by regularizing the $\mathcal{H}_2$ optimal control problem with a penalty on communication requirements in the distributed controller. In contrast to previous work [3–5], this regularization penalty reflects the fact that sparsity should be enforced in a specific set of coordinates. In [3–5], the elements of the state-feedback gain matrix were taken to represent communication links. Herein, we present a unified framework where a communication link is a linear function of the elements of the output-feedback gain matrix.

The proposed framework addresses challenges that arise in systems with invariances and symmetries, as well as consensus and synchronization networks. For example, the block diagonal structure of spatially-invariant systems in the spatial frequency domain facilitates efficient computation of the optimal centralized controllers [1]. However, since the sparsity requirements are typically expressed in the physical space, it is challenging to translate them into frequency domain specifications. Furthermore, in wide-area control of power networks [19–21], it is desired to design the controllers that respect the structure of the original system: in both open- and closed-loop networks, only relative rotor angle differences between different generators are allowed to appear. To deal with these structural requirements, we introduce a coordinate transformation to eliminate the average mode and assure stabilizability and detectability of the remaining modes. Once again, it is desired to promote sparsity of the feedback gain in physical domain and it is challenging to translate these requirements in the transformed set of coordinates.

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We leverage the alternating direction method of multipliers (ADMM) [22] to exploit the structure of the corresponding objective functions in the regularized optimal control problem. ADMM alternates between optimizing closed-loop performance and promoting sparsity of the feedback gain matrix. The sparsity promoting step in ADMM has an explicit solution and the performance optimization step is solved using Anderson-Moore and proximal gradient methods. Our framework thus allows for performance and sparsity requirements to be expressed in different set of coordinates and facilitates efficient computation of sparse static output-feedback controllers.

For undirected consensus networks, the proposed approach admits a convex characterization. Furthermore, for systems with invariances and symmetries, transform techniques are utilized to gain additional computational advantage and improve efficiency. For example, by bringing the matrices associated with a state-space representation of a spatially-invariant system into a block-diagonal form, the regularized optimal control problem amounts to easily parallelizable task of solving a sequence of smaller, fully-decoupled problems. While the computational complexity of algorithms that do not exploit spatially invariant structure increases cubicly with the number of subsystems, our algorithms exhibit a linear growth. After having identified a controller structure, the structured design step optimizes the network performance over the identified structure.

Our presentation is organized as follows. In Section 2, we provide motivating examples and formulate the generalized sparsity-promoting optimal control problem that we study in this paper. In Section 3, we identify a class of convex problems that can be cast as semidefinite programs. In Section 4, we leverage the alternating direction method of multipliers algorithm to exploit the structure of the corresponding objective functions and solve the regularized optimal control problem. In Section 5, we illustrate our developments using a synchronization network. We conclude the paper in Section 6.

2. Motivation and background

We consider a class of control problems

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\,\hat{x} + \hat{B}_1\,\hat{d} + \hat{B}_2\,\hat{u} \\ \hat{z} &= \hat{C}_1\,\hat{x} + \hat{D}\,\hat{u} \\ \hat{y} &= \hat{C}_2\,\hat{x} \\ \hat{u} &= -\hat{K}\,\hat{y} \end{aligned} \tag{1}$$

where $\hat{x}$ is the state, $\hat{d}$ and $\hat{u}$ are the disturbance and control inputs, $\hat{z}$ is the performance output, and $\hat{y}$ is the measured output. The matrices $\hat{C}_1$ and $\hat{D}$ are given by $\begin{bmatrix}\hat{Q}^{1/2} & 0\end{bmatrix}^*$ and $\begin{bmatrix}0 & \hat{R}^{1/2}\end{bmatrix}^*$ with standard assumptions on stabilizability and detectability of pairs $(\hat{A}, \hat{B}_2)$ and $(\hat{A}, \hat{Q}^{1/2})$. Here, $(\cdot)^*$ denotes complex-conjugate transpose of a given matrix. The matrices $\hat{Q} = \hat{Q}^* \succeq 0$ and $\hat{R} = \hat{R}^* \succ 0$ are the state and control performance weights, and the closed-loop system is given by

$$\begin{aligned} \dot{\hat{x}} &= (\hat{A} - \hat{B}_2\,\hat{K}\,\hat{C}_2)\hat{x} + \hat{B}_1\,\hat{d} \\ \hat{z} &= \begin{bmatrix} \hat{Q}^{1/2} \\ -\,\hat{R}^{1/2}\,\hat{K}\,\hat{C}_2 \end{bmatrix}\hat{x}. \end{aligned} \tag{2}$$

We assume that there is a stabilizing feedback gain matrix $\hat{K}$.

Our objective is to achieve a desired tradeoff between the $\mathcal{H}_2$ performance of system (2) and the sparsity of a matrix that is related to the feedback gain matrix $\hat{K}$ through a linear transformation $\mathcal{T}(\hat{K})$. To address this challenge we consider a regularized optimal control problem

$$\underset{\hat{K}}{\text{minimize}}\;\; J(\hat{K}) + \gamma\, g(\mathcal{T}(\hat{K})) \tag{3}$$

where $J(\hat{K})$ is the $\mathcal{H}_2$ norm of system (2), γ is a positive regularization parameter, and $g(\mathcal{T}(\hat{K}))$ is a sparsity-promoting regularization term (see Section 2.3 for details).

Linear transformation $\mathcal{T}(\hat{K})$ of the feedback gain $\hat{K}$ in (3) reflects the fact that *sparsity should be enforced in a specific set of coordinates*. This characterization is more general than the one considered in [3–5] where the sparsity-promoting optimal control was originally introduced and algorithms were developed. In contrast to [3–5], where it was assumed that the state-space model is given in physically meaningful coordinates, herein we only require that the states in (2) are related to these coordinates via a linear transformation $\mathcal{T}$. One such example arises in spatially invariant systems where the "spatial frequency" domain is convenient for minimizing quadratic performance objective [1], whereas sparsity requirements are naturally expressed in the physical domain. Another class of problems is given by consensus and synchronization networks where the absence of absolute measurements confines standard control-theoretic requirements to a subspace of the original state-space.

2.1. Problem formulation

As mentioned earlier, while it is convenient to formulate minimization of the quadratic performance index in terms of the feedback gain $\hat{K}$, it may be desirable to promote sparsity in a different set of coordinates. By introducing an additional optimization variable K, we bring (3) into the following form,

$$\begin{aligned} &\underset{\hat{K},\,K}{\text{minimize}} && J(\hat{K}) + \gamma\, g(K) \\ &\text{subject to} && \mathcal{T}(\hat{K}) - K = 0, \end{aligned} \tag{4a}$$

where $g(K)$ is a sparsity-promoting regularization term and $\mathcal{T}$ is a linear operator. In the $\mathcal{H}_2$ setting, $J(\hat{K})$ is given by

$$J(\hat{K}) := \begin{cases} \text{trace}\left((\hat{Q} + \hat{C}_2^*\hat{K}^*\hat{R}\hat{K}\hat{C}_2)\hat{X}\right), & \hat{K} \text{ stabilizing} \\ \infty, & \text{otherwise} \end{cases} \tag{4b}$$

where the closed-loop controllability Gramian $\hat{X}$ satisfies the Lyapunov equation

$$(\hat{A} - \hat{B}_2\hat{K}\hat{C}_2)\hat{X} + \hat{X}\,(\hat{A} - \hat{B}_2\hat{K}\hat{C}_2)^* + \hat{B}_1\hat{B}_1^* = 0. \tag{4c}$$

Clearly, for any feasible $\hat{K}$ and K, the optimal control problems (3) and (4a) are equivalent. We note that the linear constraint in (4a) is more general than the constraint considered in [3–5], where $\hat{K} - K = 0$. This introduces additional freedom in control design and broadens applicability of the developed tools.

In the set of coordinates where it is desired to promote sparsity, the closed-loop system takes the form

$$\begin{aligned} \dot{x} &= (A - B_2\,K\,C_2)\,x + B_1\,d \\ z &= \begin{bmatrix} Q^{1/2} \\ -R^{1/2}K\,C_2 \end{bmatrix} x, \end{aligned} \tag{5}$$

where $K = \mathcal{T}(\hat{K})$.

2.2. Examples

We next discuss several classes of problems that are encountered in applications. For each of these, the optimal control problem can be brought into the form (4) via a suitable change of coordinates.