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Delayed unknown input observers for discrete-time linear systems with guaranteed performance

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ABSTRACT

In this paper, we propose a state and unknown input observer for discrete-time linear systems with bounded unknown inputs and measurement disturbances. The design procedure is formulated using a set of linear matrix inequalities, and leverages delayed (or fixed-lag) estimates. The observer error states and/or user-defined performance outputs are guaranteed to operate at certain performance bounds. Furthermore, by employing sufficiently large delays, the observer is guaranteed to provide exact asymptotic state and input estimates for minimum-phase systems. We demonstrate, via numerical examples, that the proposed observer can be used for a wider class of systems than those satisfying matching conditions.

1. Introduction

The problem of estimating the states of a dynamical system in the presence of unknown disturbance inputs arises in a variety of applications. Examples include security for cyber-physical systems where the attack vector is modeled as an unknown input [1,2], networked or decentralized control where control input information at different nodes is unavailable and must be estimated [3], and fault detection and estimation in large-scale systems [4]. Estimating system states in the presence of the disturbance inputs is usually done through robust state estimators. Such formalisms include set-valued observers [5–7] or $\mathcal{H}_2/\mathcal{H}_\infty$ filtering [8–10]. For disturbances whose stochastic descriptions are known, robust state estimators such as Kalman filters [11,12] and other minimum variance filters [13–15] have been widely investigated.

For the case of completely arbitrary unknown inputs, the current literature contains a variety of unknown input observer (UIO) architectures. A discussion of strong observability and conditions for unknown input reconstruction can be found in [16] and [17]. Necessary and sufficient conditions for the existence of discrete time UIOs are proposed in [18,19]. A relaxation of these stringent conditions is discussed in [20] by allowing delays in estimation. Recent discrete-time UIO methodologies have continued to explore the systematic use of time delays, such as [21] which proposes the use of polynomial fitting to reconstruct the unknown input. Geometric conditions for observer design with delayed measurements are discussed in [22]. In [23], a delayed observer with fictitious outputs is considered that enables state and unknown input reconstruction by exploiting left-invertibility properties of the system. In [24], the dimensionality issue is improved by constructing delayed observers for a reduced-order system. An adaptation gain term is used in [25] to generate the unknown input using delayed observers. Other formulations of unknown input observers are found in [26–29].

One of the key insights established by the existing literature is that perfect asymptotic state and input estimation is possible in the presence of arbitrary unknown inputs *if and only if* the system satisfies a so-called 'minimum phase' condition. Furthermore, under this condition, real-time estimation may not be possible unless the system satisfies certain 'matching' conditions, which is the reason for introducing delays in estimation. These conditions pose certain challenges. First, one may be interested in estimating the states or inputs of non-minimum phase systems. Second, even if the system is minimum phase, the delay required to completely decouple the effects of the unknown inputs may be larger than desired. Thus, there is a need to construct observers that generate accurate estimates of plant states and unknown exogenous inputs with specified maximum bounds on estimation delays.

In this paper we address the above issues by formulating an observer that provides a guaranteed level of attenuation for *bounded* unknown inputs with any specified maximum estimation delay. Specifically, we provide sufficient conditions in the form

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of linear matrix inequalities (LMIs) for the construction of the observer gains, and compute peak-gain performance bounds on a pre-specified performance output of the observer. Additionally, we propose sufficient conditions that ensure that the unknown inputs can be reconstructed to a specified level of accuracy. Our observer generalizes existing approaches in that it achieves perfect attenuation of the unknown inputs if the system is minimum-phase and the specified delay is sufficiently large; however, when these conditions are not satisfied, the performance of our observer degrades gracefully with the magnitude of the unknown inputs as long as the system is detectable (which is a necessary condition for the construction of *any* estimator).

2. Notation

We denote by $\mathbb{R}$ the set of real numbers, $\mathbb{N}$ the set of natural numbers, and $\mathbb{R}^{n\times m}$ the set of $n \times m$ matrices for any $m, n \in \mathbb{N}$. For any vector $v \in \mathbb{R}^n$, we denote $\|v\| = \sqrt{v^\top v}$. For a sequence of vectors $\{v_k\}_{k=0}^{\infty}$, we denote $\|v\|_\infty \triangleq \sup_{k\geq 0}\|v_k\|$; consequently, we say a sequence $\{v_k\} \in \ell_\infty$ if $\|v\|_\infty < \infty$. For any matrix $P \in \mathbb{R}^{n\times n}$, we denote $P^\top$ as its transpose, and $\|P\|$ as the maximum singular value of P. For a symmetric matrix $M = M^\top$, we use the star notation to avoid rewriting symmetric terms, that is, $\begin{bmatrix} M_a & \star \\ M_b^\top & M_c \end{bmatrix} \equiv \begin{bmatrix} M_a & M_b \\ M_b^\top & M_c \end{bmatrix}$.

3. Problem statement and proposed solution

3.1. Problem statement

We consider a class of discrete-time linear systems modeled by

$$x_{k+1} = Ax_k + Bw_k \tag{1a}$$

$$y_k = Cx_k + Dw_k, \tag{1b}$$

where $x_k \in \mathbb{R}^{n_x}$ denotes the state vector at the kth sampled time, and $w_k \in \mathbb{R}^{n_w}$ denotes the vector of *exogenous* unknown inputs (e.g., disturbance inputs in the state and output vector fields, measurement noise, attack vectors, *etc.*). The measured output is denoted by $y_k \in \mathbb{R}^{n_y}$. The matrices A, B, C, D are of appropriate dimensions. The initial sample time is $k = 0$. We make the following assumptions on the class of systems considered in this paper.

Assumption 1. The unknown inputs are bounded, that is, the disturbance input sequence $\{w_k\} \in \ell_\infty$.

Note that the bounds mentioned in Assumption 1 are not necessarily known by the designer.

Assumption 2. The matrix $G \triangleq \begin{bmatrix} B^\top & D^\top \end{bmatrix}^\top$ has full column rank.

Remark 1. Assumption 2 is mild as the linearly dependent columns of G can be removed without affecting the column space through which the exogenous inputs act.

Our **objective** is to construct a robust observer that reconstructs the states x_k of the plant while attenuating the effect of the unknown exogenous input w_k. As discussed in Section 1, we will be considering observers that allow a pre-specified delay in estimation. Before we introduce the specific observer structure, it will be useful to introduce some notation. For any $\delta \in \mathbb{N}$, define

$$Y_{k:k+\delta} \triangleq \begin{bmatrix} y_k^\top & y_{k+1}^\top & \cdots & y_{k+\delta-1}^\top & y_{k+\delta}^\top \end{bmatrix}^\top. \tag{2}$$

From the dynamics (1), we obtain

$$Y_{k:k+\delta} = \Theta_\delta x_k + \Gamma_\delta W_{k:k+\delta},$$

where $W_{k:k+\delta} = \begin{bmatrix} w_k^\top & w_{k+1}^\top & \cdots & w_{k+\delta-1}^\top & w_{k+\delta}^\top \end{bmatrix}^\top$, and

$$\Theta_\delta = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\delta-1} \end{bmatrix}, \quad \Gamma_\delta = \begin{bmatrix} D & 0 & \dots & 0 \\ CB & D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{\delta-2}B & CA^{\delta-3}B & \dots & D \end{bmatrix}.$$

3.2. Proposed observer architecture

Let $\delta \in \mathbb{N}$ be a constant specifying the maximum delay (in number of time-steps) that can be tolerated for estimating the state of the system. The proposed observer has the form

$$\hat{x}_{k+1} = Q\hat{x}_k + LY_{k:k+\delta} \tag{3}$$

where $\hat{x}_k \in \mathbb{R}^{n_x}$ is an estimate of the state x_k at the kth time instant, and $Q \in \mathbb{R}^{n_x\times n_x}$, $L \in \mathbb{R}^{n_x\times(\delta+1)n_y}$ are observer gain matrices to be designed.

Note that the observer updates an estimate of the state x_k based on the measurements of the system from time-step k to $k+\delta$. When $\delta = 0$ the observer is in the form of a *predictor* (as it estimates x_{k+1} based on y_k). When $\delta = 1$, the observer estimates the state x_{k+1} using measurements up to y_{k+1}, and when $\delta > 1$, the observer is analogous to a *fixed-lag smoother* from the filtering literature [30]. Note that one can equivalently view this observer as providing an estimate of the state $x_{k-\delta+1}$ based on measurements from time-step $k - \delta$ to the current time-step k.

Delayed observers are useful in process monitoring and fault detection—in such cases, one can allow some delays in estimation if that provides a better estimate of the state in the presence of faults/attacks/disturbances; see, for example [31–34].

3.3. Observer error dynamics

Let the observer error at the kth time step be defined as $e_k = \hat{x}_k - x_k$. Then from (1), (3), the error dynamics are governed by

$$\begin{aligned} e_{k+1} &= \hat{x}_{k+1} - x_{k+1} \\ &= Qe_k + (L\Theta_\delta - A + Q)x_k + (L\Gamma_\delta - \Phi_\delta)W_{k:k+\delta} \end{aligned} \tag{4}$$

where

$$\Phi_\delta \triangleq \begin{bmatrix} B & 0 & \cdots & 0 \end{bmatrix}. \tag{5}$$

We define a **performance output**

$$z_k \triangleq He_k \tag{6}$$

where $z_k \in \mathbb{R}^{n_z}$, and $n_z \leq n_x$. This performance output is employed to select subsets/linear combinations of error states that are most crucial to the specific application, thereby requiring maximal disturbance attenuation.

Before we formally state our objective, we present the following definition from [35]:

Definition 1 (*ℓ_∞-stability with performance level γ*)**.** Consider a discrete-time error system

$$e_{k+1} = \phi(k, e_k, d_k) \tag{7a}$$

with state e_k, disturbance input sequence $\{d_k\}$, and performance output

$$z_k = \psi(e_k) \tag{7b}$$

where the input sequence $\{d_k\} \in \ell_\infty$. The system is said to be **globally uniformly ℓ_∞-stable with a specified performance level γ with respect to the disturbance input sequence $\{d_k\}$** if the following conditions are satisfied: