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A fractional representation approach to the robust regulation problem for SISO systems

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ABSTRACT

The purpose of this article is to develop a new approach to the robust regulation problem for plants which do not necessarily admit coprime factorizations. The approach is purely algebraic and allows us dealing with a very general class of systems in a unique simple framework. We formulate the famous internal model principle in a form suitable for plants defined by fractional representations which are not necessarily coprime factorizations. By using the internal model principle, we are able to give necessary and sufficient solvability conditions for the robust regulation problem and to parameterize all robustly regulating controllers.

1. Introduction

Robustness of controllers is of fundamental importance since it allows them to work under uncertain conditions. Regulating controllers can asymptotically track a given reference signal. Robustness means that the controller remains regulating despite small perturbations of the system. For example, modeling errors, model simplifications and attrition of components in a real world application can be seen as perturbations of the system. The robust regulation problem is to find a robustly regulating controller.

Robust regulation of finite-dimensional plants is well-understood [1–3]. The finite-dimensional theory has been generalized to infinite-dimensional plants and signals by several authors. See, for instance, [4–12] and the references therein. One of the most fundamental results of robust regulation is the internal model principle, which states that any robustly regulating controller contains a suitably reduplicated model of the dynamics to be tracked.

In the frequency domain, the robust regulation problem is an algebraic problem. Vidyasagar formulated and solved it by using coprime factorizations over the ring of stable rational transfer functions [3]. Vidyasagar's results state the internal model principle, give a necessary and sufficient solvability condition of the problem, and parameterize all robustly regulating controllers in a remarkably simple form. These results have been generalized to fields of fractions over rings suitable for distributed parameter systems and/or infinite-dimensional reference and disturbance signals [4,5,7,9,10,12]. The common feature of the results is that they require the existence of coprime factorizations. This is problematic since all plants do not possess coprime factorizations [13,14], or their existence is not known [9,15].

In this paper, we develop robust regulation theory of single-input single-output (SISO) plants based on stabilizability results of [16]. The advantage of the theory presented in [16] is that it uses no coprime factorizations and allows us to develop theory with very few assumptions. We only need to define a commutative ring A of stable elements with a unit and having no zero divisors to start with. The plants are just elements in the field of fractions over A. This makes the theory applicable in several different classes of infinite-dimensional systems, for instance in those of [9,17]. From the theoretic point of view, the choice of A is irrelevant, but when applying the results, the choice of A depends naturally on the problem at hand. Examples of rings motivated by systems theoretic applications involve H^∞ and the Callier–Desoer algebra where all stabilizable plants have coprime factorizations, $A := \mathbb{R}[x^2, x^3]$ of Example 5.1 with plants without weakly coprime factorizations, and $\mathbf{P}$ of [9], for which the existence of (weakly) coprime factorizations of stabilizable transfer functions is not known.

The abstract algebraic approach to robust regulation has received only little attention this far. In the last chapter of his book [3], Vidyasagar discussed the generalization of finite-dimensional stabilization and regulation theory to infinite-dimensional systems. Unfortunately, the part concerning robust regulation uses coprime factorizations and therefore is not applicable for general rings. The same is true for the theory developed in [10]. In addition, both of the above references use topological notions in the study of robustness. It is possible to do without by defining the robustly regulating controllers so that they are

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exactly the ones that are regulating for every plant they stabilize. This definition splits the robust regulation problem into two parts: robust regulation that involves constructing an internal model into the controller and robust stabilization that involves the topological aspects of the problem. In this article, we focus on the former. Robustness of stability is well-understood in many physically interesting algebraic structures [3,18] as well as in the abstract setting [19,20].

By using the fractional representation approach, we generalize the theory of [3] to the plants which do not necessarily possess coprime factorizations. The main contributions of this article are:

- We give a reformulation of the internal model principle without using coprime factorizations.
- We give a checkable necessary and sufficient condition for solvability of the robust regulation problem.
- We parameterize all robustly regulating controllers for signal generators with a weakly coprime factorization.

The internal model principle and the solvability condition can be found in the preliminary version [21] of this article. However, in this article, we require only weakly coprime factorizations instead of coprime factorizations, which extends some of the results of [21]. Theorem 4.6 and Corollary 4.8, which give a parametrization of all robustly regulating controllers, are new. We formulate the results of this paper using fractional representations. For fractional ideal approach, see [21].

The remaining part of the paper is organized as follows. Notations, preliminary results, and the problem formulation are given in Section 2. The internal model principle is considered in Section 3. Section 4 contains solvability considerations and, by using the results of the section, we are able to give a parametrization of all robustly regulating controllers. In Section 5, we illustrate the theoretical results by examples. Finally, the concluding remarks are made in Section 6.

2. The problem formulation

Let A be an *integral domain*, namely a commutative ring with a unit element 1 and without zero divisors [22]. We denote by $A^{l\times m}$ the A-module of $l \times m$ matrices with entries in A and by

$$Q(A) := \left\{ \frac{n}{d} \mid 0 \neq d,\ n \in A \right\}$$

the field of fractions of A.

Definition 2.1.

1. An element $h \in Q(A)$ (resp., a matrix $H \in Q(A)^{l\times m}$) is said to be *stable* if we have $h \in A$ (resp., $H \in A^{l\times m}$) and *unstable* otherwise.
2. A controller $c \in Q(A)$ *stabilizes* $p \in Q(A)$ if the closed loop system of Fig. 1 from $(y_r \quad d)^T$ to $(e \quad u)^T$ given by

$$H(p, c) := \begin{pmatrix} \frac{1}{1-p\,c} & \frac{p}{1-p\,c} \\ \frac{c}{1-p\,c} & \frac{1}{1-p\,c} \end{pmatrix}$$

is stable, i.e., if we have $H(p, c) \in A^{2\times 2}$.

Let $\mathrm{Stab}(p)$ be the set of all the stabilizing controllers of p. Note that $c \in \mathrm{Stab}(p)$ is equivalent to $p \in \mathrm{Stab}(c)$.

Definition 2.2. Let $\Theta \in Q(A)$. Then, we have:

1. *A fractional representation* of Θ is defined by $\Theta = \frac{\gamma}{\theta}$, where $0 \neq \theta, \gamma \in A$.
2. A fractional representation $\Theta = \frac{\gamma}{\theta}$ is called *a coprime factorization* if there exist $\alpha,\ \beta \in A$ such that $\alpha\,\gamma - \beta\,\theta = 1$.
3. A fractional representation $\Theta = \frac{\gamma}{\theta}$ is called *a weakly coprime factorization* if we have:

$$\forall\, k \in Q(A) : k\,\gamma,\ k\,\theta \in A \Longrightarrow k \in A.$$

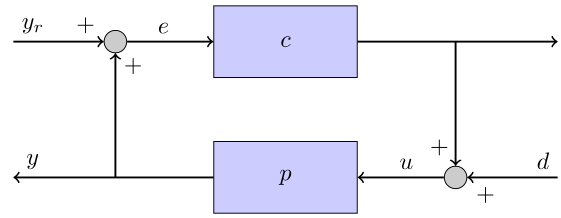

Fig. 1. The control configuration.

The approach developed in this article is based on the stabilizability results of [16]. The following theorem combines Theorems 1 and 2 of [16].

Theorem 2.3. *The plant p is stabilizable if and only if there exist $a,\ b \in A$ such that:*

$$\begin{cases} a - p\,b = 1, \\ p\,a \in A. \end{cases} \tag{1}$$

Moreover, a controller c stabilizes p if and only if it is of the form $c = \frac{b}{a}$, where $0 \neq a, b \in A$ satisfy (1). In this case, we have that $a = (1 - p\,c)^{-1}$ and $b = c\,(1 - p\,c)^{-1}$.

If $0 \neq a, b \in A$ satisfy (1), then all the stabilizing controllers of p are parametrized by

$$c(q_1, q_2) := \frac{b + q_1\,a^2 + q_2\,b^2}{a + q_1\,p\,a^2 + q_2\,p\,b^2}, \tag{2}$$

where $q_1,\ q_2 \in A$ are such that the denominator of (2) does not vanish.

We make a standing assumption that all the reference and disturbance signals are generated by a fixed signal generator $\Theta \in Q(A)$, i.e., the reference and disturbance signals are of the form:

$$y_r := \Theta\,y_0, \quad d := \Theta\,d_0, \quad y_0,\ d_0 \in A.$$

Definition 2.4.

1. We say that a controller c is *regulating* p with the signal generator Θ if

$$e = \begin{pmatrix} \frac{1}{1-p\,c} & \frac{p}{1-p\,c} \end{pmatrix} \Theta \begin{pmatrix} y_0 \\ d_0 \end{pmatrix} \in A,$$

for all $y_0,\ d_0 \in A$, or equivalently if we have:

$$\Theta \begin{pmatrix} \frac{1}{1-p\,c} & \frac{p}{1-p\,c} \end{pmatrix} \in A^{1\times 2}. \tag{3}$$

2. A controller c is called *robustly regulating* with the signal generator Θ if we have:
   1. c stabilizes p, i.e., $c \in \mathrm{Stab}(p)$.
   2. c regulates every plant it stabilizes, i.e., if for all $p' \in \mathrm{Stab}(c)$, we then have:

$$\Theta \begin{pmatrix} \frac{1}{1-p'\,c} & \frac{p'}{1-p'\,c} \end{pmatrix} \in A^{1\times 2}.$$

The robust regulation problem is the problem of finding a robustly regulating controller.