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Existence and optimality conditions for relaxed mean-field stochastic control problems[☆]

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ABSTRACT

We consider optimal control problems for systems governed by mean-field stochastic differential equations, where the control enters both the drift and the diffusion coefficient. We study the relaxed model, in which admissible controls are measure-valued processes and the relaxed state process is driven by an orthogonal martingale measure, whose covariance measure is the relaxed control. This is a natural extension of the original strict control problem, for which we prove the existence of an optimal control. Then, we derive optimality necessary conditions for this problem, in terms of two adjoint processes extending the known results to the case of relaxed controls.

1. Introduction

In this paper, we deal with optimal control of systems driven by mean-field stochastic differential equations (MFSDE), where the coefficients depend not only on the state but also on its distribution. This mean-field equation, represents in some sense the average behavior of an infinite number of particles, see [1,2] for details. Since the earlier papers [3,4], mean-field control theory has raised a lot of interest, motivated by applications to various fields such as game theory, mathematical finance, communications networks, management of oil resources. Mean-field control problems occur in many applications, such as in a continuous-time Markowitz's mean-variance portfolio selection model where the variance term involves a quadratic function of the expectation. The inclusion of this mean-field terms in the coefficients introduces time inconsistency, leading to the failure of Bellman principle. For this kind of problems, the stochastic maximum principle, provides a powerful tool to solve them, see [5,6] and the references therein. The first objective of the present paper is to investigate the problem of existence of an optimal control. It is well known that in the absence of convexity assumptions, this problem has no optimal solution. Therefore it is natural to embed the set of strict controls into a wider class of measure valued controls, enjoying good compactness properties, called relaxed controls. We show that the right state process associated with a relaxed control, satisfies a MFSDE driven by an orthogonal martingale measure rather than a Brownian motion. For this model, we prove that the strict and relaxed control problems have the same value function and that an optimal relaxed control exists. Our result extends in particular [7–10] to mean field controls and [11] to the case of a MFSDE with a controlled diffusion coefficient. The proof is based on tightness properties of the underlying processes and the Skorokhod selection theorem. In a second step, we establish necessary conditions for optimality in the form of a relaxed stochastic maximum principle, obtained via the first and second order adjoint processes. This result generalizes Peng's stochastic maximum principle [12], to mean field control problems and [5] to relaxed controls. The other advantage is that our maximum principle applies to a natural class of controls, which is the closure of the class of strict controls, for which we have existence of an optimal control. The proof of the main result is based on the approximation of the relaxed control problem by a sequence of strict control problems. Then Ekeland's variational principle is applied to get necessary conditions of near-optimality, for the sequence of near optimal strict controls. The result is obtained by a passage to the limit in the state equation as well as in the adjoint processes. The resulting first and second order adjoint processes are solutions of linear BSDEs driven by a Brownian motion and an orthogonal square integrable martingale. Moreover, our result is given via an approximation procedure, so that it could be convenient for numerical computation.

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2. Assumptions and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space, equipped with a filtration $(\mathcal{F}_t)$, satisfying the usual conditions and (W_t) a $(\mathcal{F}_t, P)$-Brownian motion. Let $\mathbb{A}$ be some compact metric space called the action space. A strict control (u_t) is a measurable, $\mathcal{F}_t$-adapted process with values in the action space $\mathbb{A}$. We denote $\mathcal{U}_{ad}$ the space of strict controls.

The state process corresponding to a strict control is the unique solution, of the mean-field stochastic differential equations (MFSDE)

$$dX_t = b(t, X_t, E(X_t), u_t)dt + \sigma(t, X_t, E(X_t), u_t)dW_t;\ X_0 = x \qquad (2.1)$$

and the corresponding cost functional is given by

$$J(u) = E\left(\int_0^T h(t, X_t), E(X_t), u_t\right) dt + g(X_T, E(X_T)).$$

The coefficients of the state equation as well as of the cost functional are of mean-field type, in the sense that they depend not only on the state process, but also on its marginal law, through its expectation.

The objective is to minimize $J(u)$ over the space $\mathcal{U}_{ad}$, that is to find $u^* \in \mathcal{U}_{ad}$ such that $J(u^*) = \inf\{J(u), u \in \mathcal{U}_{ad}\}$.

Let us consider the following assumptions which will be used in different combinations throughout the paper.

$(\mathbf{H_1})$ $b : [0, T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{A} \longrightarrow \mathbb{R}, \sigma : [0, T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{A} \longrightarrow \mathbb{R}$ are bounded continuous functions such that $b(t, ., ., a)$ and $\sigma(t, ., ., a)$ are Lipschitz continuous in (x, y).

$(\mathbf{H_2})$ $h : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{A} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, are bounded continuous functions such that $h(t, ., ., a)$ and $g(., .)$ are Lipschitz continuous in (x, y).

$(\mathbf{H_3})$ $b(t, ., ., a), \sigma(t, ., ., a), h(t, ., ., a)$ and $g(., .)$ are twice continuously differentiable with respect to (x, y), and their derivatives are bounded and continuous in (x, y, a).

Without loss of generality, the coefficients are assumed to be one dimensional as in [5], to avoid heavy notations in the definition of adjoint processes.

Under assumption $(\mathbf{H_1})$, according to [1] Prop.1.2, for each $u \in \mathcal{U}_{ad}$ the MFSDE(2.1) has a unique strong solution, such that for every $p > 0$ we have $E(|X_t|^p) < +\infty$. Moreover the cost functional is well defined.

3. The relaxed control problem

3.1. The space of relaxed controls

As it is well known in control theory, in the absence of convexity conditions, an optimal control may fail to exist in the set $\mathcal{U}_{ad}$ of strict controls (see e.g. [9]). This suggests that the set of strict controls is too narrow and should be embedded into a wider class of relaxed controls, with nice compactness properties. For the relaxed model, to be a true extension of the original control problem, the following both conditions must be satisfied:

(i) The value functions of the original and the relaxed control problems must be equal.

(ii) The relaxed control problem must have an optimal solution.

The idea of relaxed control is to replace the $\mathbb{A}$-valued process (u_t) with a $\mathbb{P}(\mathbb{A})$-valued process (μ_t), where $\mathbb{P}(\mathbb{A})$ is the space of probability measures equipped with the topology of weak convergence. Then (μ_t) may be identified as a random product measure on $[0, T] \times \mathbb{A}$, whose projection on $[0, T]$ coincides with Lebesgue measure. Let $\mathbb{V}$ be the set of product measures μ on $[0, T] \times \mathbb{A}$ whose projection on $[0, T]$ coincides with the Lebesgue measure dt. It is clear that every μ in $\mathbb{V}$ may be disintegrated as $\mu = dt.\mu_t(da)$, where $\mu_t(da)$ is a transition probability. The elements of $\mathbb{V}$ are called Young measures in deterministic theory. $\mathbb{V}$ as a closed subspace of the space of positive Radon measures $\mathbb{M}_+([0, T] \times \mathbb{A})$ is compact for the topology of weak convergence. In fact it can be proved that it is compact also for the topology of stable convergence, where test functions are measurable, bounded functions $f(t, a)$ continuous in a, see [8] for further details.

Definition 3.1. A relaxed control on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a random variable $\mu = dt.\mu_t(da)$ with values in $\mathbb{V}$, such that $\mu_t(da)$ is progressively measurable with respect to $(\mathcal{F}_t)$ and such that for each t, $1_{(0,t]}.\mu$ is $\mathcal{F}_t$-measurable.

Remark 3.2. The set $\mathcal{U}_{ad}$ of strict controls is embedded into the set of relaxed controls by identifying u_t with $dt\delta_{u_t}(da)$.

3.2. The relaxed state equation

The question now is to define the natural state process associated to a relaxed control. In deterministic control or in the stochastic theory where only the drift is controlled, one has just to replace in Eq. (2.1) the drift by the same drift integrated against the relaxed control. Now we are in a situation where both the drift and the diffusion coefficient are controlled. Following [1] Prop. 1.10, the existence of a weak solution of Eq. (2.1) associated with a strict control u is equivalent to the existence of a solution for the non linear martingale problem:

$$f(X_t) - f(X_0) - \int_0^t L^{P_{X_s}} f(s, X_s, u_s)\, ds \text{ is a } P\text{-martingale},$$

for every $f \in C_b^2$, *for each* $t > 0$, *where L is the infinitesimal generator* associated with Eq. (2.1),

$$L^{\nu} f(t, x, a) = \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2}(t, x, a) + b\frac{\partial f}{\partial x}(t, x, a),$$

$b = b(t, x, \langle y, \nu\rangle, a)$ and $\sigma^2 = \sigma^2(t, x, \langle y, \nu\rangle, a)$ where $\nu \in \mathbb{P}_1(\mathbb{R})$, the space of probability measures on $\mathbb{R}$.

Therefore, the natural relaxed martingale problem associated to a relaxed control is defined as follows:

$$f(X_t) - f(X_0) - \int_0^t \int_A L^{P_{X_s}} f(s, X_s, a)\, \mu_s(da)ds$$

is a P-martingale *for each* $f \in C_b^2$, *for each* $t > 0$.

The following theorem gives a pathwise representation of the solution of the relaxed martingale problem, in terms of a mean-field stochastic differential equation driven by an orthogonal martingale measure.

Theorem 3.3. (1) *Let P be a solution of the relaxed martingale problem. Then P is the law of an adapted, continuous process X defined on an extension of the space* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, *which is a solution of the following MFSDE:*

$$dX_t = \int_{\mathbb{A}} b(t, X_t, E(X_t), a)\, \mu_t(da)dt + \int_{\mathbb{A}} \sigma(t, X_t, E(X_t), a)\, M(da, dt);\ X_0 = x \qquad (3.1)$$

where M is an orthogonal continuous martingale measure, with intensity $dt\mu_t(da)$.

(2) *If the coefficients b and σ are Lipschitz in x, y, uniformly in t and a,* Eq. (3.1) *has a unique pathwise solution.*

Proof. (1)The proof is based essentially on the same arguments as in [13], Theorem IV-2 and [1], Prop. 1.10.

(2) Since the coefficients are Lipschitz continuous, then following the same steps as in [1,13], it is not difficult to prove that Eq. (3.1) has a unique solution such that for every $p > 0$ we have $E(\sup_{t\in[0,T]} |X_t|^p) < +\infty$. ■