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A geodesic feedback law to decouple the full and reduced attitude ☆

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ABSTRACT

This paper presents a novel approach to the problem of almost global attitude stabilization. The reduced attitude is steered along a geodesic path on the $n-1$-sphere. Meanwhile, the full attitude is stabilized on $\mathsf{SO}(n)$. This action, essentially two maneuvers in sequel, is fused into one smooth motion. Our algorithm is useful in applications where stabilization of the reduced attitude takes precedence over stabilization of the full attitude. A two parameter feedback gain affords further trade-offs between the full and reduced attitude convergence speed. The closed loop kinematics on $\mathsf{SO}(3)$ are solved for the states as functions of time and the initial conditions, providing precise knowledge of the transient dynamics. The exact solutions also help us to characterize the asymptotic behavior of the system such as establishing the region of attraction by straightforward evaluation of limits. The geometric flavor of these ideas is illustrated by a numerical example.

1. Introduction

The attitude tracking problem for a rigid-body is well-known in the literature. It is interesting from a theoretical point of view due to the nonlinear state equations and the topology of the underlying state space $\mathsf{SO}(3)$. Application oriented approaches to attitude control often make use of parameterizations such as Euler angles or unit quaternions to represent $\mathsf{SO}(3)$. The choice of parametrization is not without importance since it may affect the limits of control performance [1–3]. An often cited result states that global stability cannot be achieved on $\mathsf{SO}(3)$ by means of a continuous, time-invariant feedback [3]. It is however possible to achieve almost global asymptotic stability through continuous time-invariant feedback [2,4], almost semi-global stability [5], or global stability by means of a hybrid control approach [6]. These subjects have also been studied with regards to the reduced attitude, *i.e.*, on the 2-sphere [2,7]. The problem of pose control on $\mathsf{SE}(3)$ is strongly related to the aforementioned problems. Many of the previously referenced results can be combined with position control algorithms in an inner-and-outer-loop configuration to achieve pose stabilization [8].

Like [2,4–6], this paper provides a novel approach to the attitude stabilization problem. The generalized full attitude is stabilized on $\mathsf{SO}(n)$. Meanwhile, the generalized reduced attitude is steered along a geodesic path on the $(n-1)$-sphere. The motion of the reduced attitude is decoupled from the remaining degree of freedom of the full attitude but not vice versa. An action consisting of two sequential maneuvers is thus fused into one smooth motion. This algorithm is of use in applications where the stabilization of the reduced attitude takes precedence over that of the full attitude. A two parameter feedback gain affords further trade-offs regarding the full and reduced attitude convergence speed. The kinematic model is suited for applications in the field of visual servo control [9,10]. Consider a camera that is tracking an object. The goal is to keep the camera pointing towards the object whereas the roll angle is of secondary importance. The proposed algorithm solves this problem by steering the principal axis directly towards the object while simultaneously stabilizing the roll angles without resorting to a non-smooth control consisting of two separate motions.

While literature on the kinematics and dynamics of n-dimensional rigid-bodies (*e.g.*, [11]) may primarily be theoretically motivated, the developments also provide a unified framework for the cases of $n \in \{2, 3\}$. The generalized reduced attitude encompasses all orientations in physical space: the heading on a circle, the reduced attitude on the sphere, and the unit quaternions on the 3-sphere. Relevant literature includes works concerning stabilization [12], synchronization [13], and estimation [14] on $\mathsf{SO}(n)$. It also includes the previous work [15,16] of the authors. Note that work on $\mathsf{SO}(n)$ for $n \geq 4$ is not only of theoretical concern; it also finds applications in the visualization of high-dimensional data [17].

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Exact solutions to a closed-loop system yields insights into both its transient and asymptotic behaviors and may therefore be of value in applications. The literature on exact solutions to attitude dynamics may, roughly speaking, be divided into two separate categories. First, there are a number of works where the exact solutions are obtained during the control design process, *e.g.*, using exact linearization [18], optimal control design techniques such as the Pontryagin maximum principle [19], or in the process of building an attitude observer [20]. Second, there are studies of the equations defining rigid-body dynamics under a set of specific assumptions whereby the exact solutions become one of the main results [21–23]. This paper belongs to the second category. The closed-loop kinematics on $\mathsf{SO}(3)$ are solved for the states as functions of time and the initial conditions, providing precise knowledge of the workings of the transient dynamics.

Recent work on the problem of finding exact solutions to closed-loop systems on $\mathsf{SO}(n)$ includes [15,16]. Related but somewhat different problems are addressed in [21–23]. Earlier work [24] by the authors is strongly related but also underdeveloped; its scope is limited to the case of SO(3). This paper concerns a generalization of the equations studied in [16,24]. The results of [16] is also generalized in [15], partly towards application in model-predictive control and sampled control systems and without focus on the behavior of the reduced attitude. The work [25] addresses the problem of continuous actuation under discrete-time sampling. The exact solutions provide an alternative to the zero-order hold technique. The algorithm alternates in a fashion that is continuous in time between the closed-loop and open-loop versions of a single control law. The feedback law proposed in this paper can also be used in such applications by virtue of the exact solutions.

2. Preliminaries

Let $\mathbf{A},\mathbf{B}\in\mathbb{C}^{n\times n}$. The spectrum of $\mathbf{A}$ is written as $\sigma(\mathbf{A})$. Denote the transpose of $\mathbf{A}$ by $\mathbf{A}^\top$ and the complex conjugate by $\mathbf{A}^*$. The inner product is defined by $\langle\mathbf{A},\mathbf{B}\rangle=\operatorname{tr}(\mathbf{A}^\top\mathbf{B})$ and the Frobenius norm by $\|\mathbf{A}\|_F=\langle\mathbf{A},\mathbf{A}\rangle^{\frac{1}{2}}$. The outer product of two vectors $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ is defined as $\mathbf{x}\otimes\mathbf{y}=\mathbf{x}\mathbf{y}^\top$. The commutator of two matrices is $[\mathbf{A},\mathbf{B}]=\mathbf{AB}-\mathbf{BA}$ and the anti-commutator is $\{\mathbf{A},\mathbf{B}\}=\mathbf{AB}+\mathbf{BA}$.

The special orthogonal group is $\mathsf{SO}(n)=\{\mathbf{R}\in\mathbb{R}^{n\times n}\mid\mathbf{R}^{-1}=\mathbf{R}^\top,\det\mathbf{R}=1\}$. The special orthogonal Lie algebra is $\mathsf{so}(n)=\{\mathbf{S}\in\mathbb{R}^{n\times n}\mid\mathbf{S}^\top=-\mathbf{S}\}$. The n-sphere is $\mathcal{S}^n=\{\mathbf{x}\in\mathbb{R}^{n+1}\mid\|\mathbf{x}\|=1\}$. The geodesic distance between $\mathbf{x},\mathbf{y}\in\mathcal{S}^n$ is given by $\vartheta(\mathbf{x},\mathbf{y})=\arccos\langle\mathbf{x},\mathbf{y}\rangle$. An almost globally asymptotically stable equilibrium is stable and attractive from all initial conditions in the state space except for a set of zero measure. The terms attitude stabilization, reduced attitude stabilization, and geodesic path refer to the stabilization problem on $\mathsf{SO}(n)$, the n-sphere, and curves that are geodesic up to parametrization respectively.

Real matrix valued, real matrix variable hyperbolic functions are defined by means of the matrix exponential, *e.g.*, $\cosh:\mathbb{R}^{n\times n}\to\mathsf{GL}(n)$ is given by $\cosh(\mathbf{A})=\frac{1}{2}[\exp(\mathbf{A})+\exp(-\mathbf{A})]$ for all $\mathbf{A}\in\mathbb{R}^{n\times n}$. Let $\operatorname{Log}:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ denote the principal logarithm, *i.e.*, $\operatorname{Log}z=\log r+i\vartheta$, where $z=re^{i\vartheta}$ and $\vartheta\in(-\pi,\pi]$. Let $\operatorname{Atanh}:\mathbb{C}\setminus\{-1,1\}\to\mathbb{C}$ denote the principal inverse hyperbolic tangent, *i.e.*, $\operatorname{Atanh}z=\frac{1}{2}[\operatorname{Log}(1+z)-\operatorname{Log}(1-z)]$. Note that $\tanh:\mathbb{C}\setminus\{-1,1\}\to\mathbb{C}$ satisfies $\tanh\operatorname{Atanh}z=z$ for all $z\in\mathbb{C}\setminus\{-1,1\}$. Extend these definitions to the extended real number line $\mathbb{R}\cup\{-\infty,\infty\}$ and the Riemann sphere $\mathbb{C}\cup\{\infty\}$ by letting $\log 0=-\infty$, $\operatorname{Atanh}1=\infty$, $\tanh\infty=1$ *etc.* [26].

3. Problem description

3.1. Stabilization and tracking

The orientation or attitude of a rigid body is represented by a rotation matrix that transforms the body fixed frame into a given inertial fixed frame. Let $\mathbf{X}\in\mathsf{SO}(3)$ denote this rotation matrix. The kinematics of a rigid body dictates that $\dot{\mathbf{X}}=\boldsymbol{\Omega}\mathbf{X}$, where $\boldsymbol{\Omega}\in\mathsf{so}(3)$ is a skew-symmetric matrix representing the angular velocity vector of the rigid body. The attitude stabilization problem is the problem of designing a feedback law that stabilizes a desired frame $\mathbf{X}_d$ which without loss of generality can be taken to be the identity matrix.

The attitude tracking problem concerns the design of an $\boldsymbol{\Omega}$ that rotates $\mathbf{X}$ into a desired moving frame $\mathbf{X}_d\in\mathsf{SO}(3)$. Assume that $\mathbf{X}_d$ is generated by $\dot{\mathbf{X}}_d=\boldsymbol{\Omega}_d\mathbf{X}_d$, where $\boldsymbol{\Omega}_d\in\mathsf{so}(3)$ is known. Furthermore assume that the relative rotation error $\mathbf{R}=\mathbf{X}_d^\top\mathbf{X}\in\mathsf{SO}(3)$ is known to the feedback algorithm. Note that rotating $\mathbf{X}$ into $\mathbf{X}_d$ is equivalent to rotating $\mathbf{R}$ into $\mathbf{I}$. Moreover,

$$\begin{aligned}\dot{\mathbf{R}}&=\dot{\mathbf{X}}_d^\top\mathbf{X}+\mathbf{X}_d^\top\dot{\mathbf{X}}=(\boldsymbol{\Omega}_d\mathbf{X}_d)^\top\mathbf{X}+\mathbf{X}_d^\top\boldsymbol{\Omega}\mathbf{X}\\&=-\mathbf{X}_d^\top\boldsymbol{\Omega}_d\mathbf{X}_d\mathbf{R}+\mathbf{X}_d^\top\boldsymbol{\Omega}\mathbf{X}_d\mathbf{R}=\mathbf{X}_d^\top(-\boldsymbol{\Omega}_d+\boldsymbol{\Omega})\mathbf{X}_d\mathbf{R}=\mathbf{U}\mathbf{R},\end{aligned}\qquad(1)$$

where $\mathbf{U}=\mathbf{X}_d^\top(-\boldsymbol{\Omega}_d+\boldsymbol{\Omega})\mathbf{X}_d\in\mathsf{so}(3)$. The kinematic level attitude tracking problem in the case of known $\mathbf{R}_d$, $\boldsymbol{\Omega}_d$ can hence be reduced to the attitude stabilization problem. It is also clear that attitude stabilization is a special case of attitude tracking.

From a mathematical perspective it is appealing to strive for generalization. Consider the evolution of a positively oriented n-dimensional orthogonal frame represented by $\mathbf{R}\in\mathsf{SO}(n)$. The dynamics are given by

$$\dot{\mathbf{R}}=\mathbf{U}\mathbf{R},\qquad(2)$$

where $\mathbf{U}\in\mathsf{so}(n)$. The kinematic level generalized attitude stabilization problem concerns the design of an $\mathbf{U}$ that stabilizes the identity matrix on $\mathsf{SO}(n)$. It is assumed that $\mathbf{R}$ can be actuated along any direction of its tangent plane at the identity $\mathsf{T}_{\mathbf{I}}\mathsf{SO}(n)=\mathsf{so}(n)$. Note that $\mathsf{SO}(n)$ is invariant under the kinematics (2), *i.e.*, any solution $\mathbf{R}(t)$ to (2) that satisfies $\mathbf{R}(0)=\mathbf{R}_0\in\mathsf{SO}(n)$ remains in $\mathsf{SO}(n)$ for all $t\in[0,\infty)$.

3.2. The reduced attitude

It is sometimes preferable to only consider $n-1$ of the $\frac{1}{2}n(n-1)$ degrees of freedom on $\mathsf{SO}(n)$. In the case of $\mathsf{SO}(3)$, these correspond to the reduced attitude [2]. The reduced attitude consists of the points on the unit sphere $\mathcal{S}^2\simeq\mathsf{SO}(3)/\mathsf{SO}(2)$. It formalizes the notion of pointing orientations such as the two degrees of rotational freedom possessed by objects with cylindrical symmetry. The reduced attitude is also employed in redundant tasks like robotic spray painting and welding that only require the utilization of two of the usual three degrees of rotational freedom in physical space [27].

Reduced attitude control by means of kinematic actuation is a special case of control on the unit n-sphere, $\mathcal{S}^n=\{\mathbf{x}\in\mathbb{R}^{n+1}\mid\|\mathbf{x}\|=1\}$. The generalized reduced attitude can be used to model all physical rotations. The heading of a two-dimensional rigid-body is an element of $\mathcal{S}^1$, the pointing direction of a cylindrical rigid-body is an element of $\mathcal{S}^2$, and the full attitude can be parametrized by $\mathcal{S}^3$ through a composition of two maps via the unit quaternions $\mathcal{S}^0(\mathbb{H})=\{q\in\mathbb{H}\mid|q|=1\}$.

Let $\mathbf{e}_1\in\mathcal{S}^{n-1}$ be a vector expressed in the body-fixed frame of an n-dimensional rigid body. The reduced attitude $\mathbf{x}\in\mathcal{S}^{n-1}$ is defined as the inertial frame coordinates of $\mathbf{e}_1$, *i.e.*, $\mathbf{x}=\mathbf{X}\mathbf{e}_1$. The reduced attitude stabilization problem is solved by a feedback algorithm that can turn $\mathbf{x}$ into any desired value $\mathbf{x}_d\in\mathcal{S}^{n-1}$. Note that $\dot{\mathbf{x}}=\dot{\mathbf{X}}\mathbf{e}_1=\boldsymbol{\Omega}\mathbf{X}\mathbf{e}_1=\boldsymbol{\Omega}\mathbf{x}$. Assume that $\mathbf{x}_d=\mathbf{X}_d\mathbf{e}_1$ satisfies $\dot{\mathbf{x}}_d=\boldsymbol{\Omega}_d\mathbf{x}_d$. Set $\mathbf{r}=\mathbf{X}_d^\top\mathbf{x}=\mathbf{X}_d^\top\mathbf{X}\mathbf{e}_1=\mathbf{R}\mathbf{e}_1$. Turning $\mathbf{x}$ into $\mathbf{x}_d$ is equivalent to turning $\mathbf{r}$ into $\mathbf{e}_1$. Moreover, $\dot{\mathbf{r}}=\dot{\mathbf{R}}\mathbf{e}_1=\mathbf{U}\mathbf{R}\mathbf{e}_1=\mathbf{U}\mathbf{r}$, where $\mathbf{U}=\mathbf{X}_d^\top(-\boldsymbol{\Omega}_d+\boldsymbol{\Omega})\mathbf{X}_d\in\mathsf{so}(n)$, like in the $\mathsf{SO}(3)$ case. The evolution of $\mathbf{r}$ is controllable on $\mathcal{S}^{n-1}$ [28].