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On the fixed controllable subspace in linear structured systems

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A R T I C L E I N F O

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A B S T R A C T

In this paper we consider interconnected networks that are described by means of structured linear systems with state and control variables. We represent these systems, whose matrices contain fixed zeros and free parameters, by means of directed graphs and study questions concerning controllability and the controllable subspace.

We show in this paper that the controllable subspace can have a part that will be present for almost all values of the free parameters. It actually is a subspace of the controllable subspace and will be referred to as the fixed controllable subspace. The subspace can then be seen as a kind of robustly controllable part of the system. Indeed, it is a subspace in the state space with the generic property that states in it can be steered in an arbitrary way.

We derive a characterization of the fixed controllable subspace using the graph representation. The obtained characterization makes use of well-known algorithms from optimization and networks theory. To get some more insight in the components in the fixed part, we also give a representation of the structured linear systems by means of bipartite graphs. Using the Dulmage–Mendelsohn decomposition, we are able to decompose our structured systems in such a way that in some special cases, the fixed controllable subspace can be obtained directly from the decomposition.

1. Introduction and motivation

Our everyday life is now related with complex networks. Examples of such complex networks appear in biology, genetics, social networks, large communication or energy networks [1]. The recent interest of the network community for some concepts of control theory has raised a number of new control problems [2–4]. The controllability of networks has been studied in the framework of structured systems. This framework is well fitted for this type of study because it can take into account loosely defined large scale systems, and it is based on a graph representation of their structure. It is interesting to note that the problem of minimizing the action on a network to control it was converted in control theory into a very nice Minimal Controllability Problem which was solved very recently [5–9]. Besides these qualitative approaches, an energy minimization paradigm has also been recently explored, see for example [10,11].

This scientific context has renewed the interest for linear structured system theory and raised some interesting new problems in relation with controllability. Linear structured systems are dynamical systems for which the entries of the classical (A, B) matrices are either zeros or independent parameters. For such systems one can study generic (or structural) properties, *i.e.*, properties which are true for almost any value of the nonzero parameters.

It happens that a lot of generic properties of structured systems can be related with a directed graph which is naturally associated with the structured system. In his fundamental paper [12], Lin laid the foundation of structured systems theory and gave a nice characterization of structural controllability in terms of particular graph objects, called cacti. The structural controllability conditions have been refined by several authors, see for example [13,14], and are now well understood.

When the conditions for structural controllability are not satisfied, it is important to quantify to which extend the system is controllable. One way to do that is to consider the dimension of the controllable subspace. The generic dimension of the controllable subspace in graph-theoretic terms was given first in [15] and expanded in [16]. Notice that this dimension was used in network theory as a measure of the importance of a particular node of driving the network, it is then called the control centrality of the node [2,17,18].

Our observation is that the dimension of the controllable subspace is certainly a question of interest but, since the controllable

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subspace is varying with the system parameters, the knowledge of this dimension does not say much about the possibility to reach a given state by a suitable control.

This is why we introduce the notion of the *fixed controllable subspace*.[1] This subspace contains all the fixed directions of the state space that can be reached and covered by a control, for almost any value of the system parameters. We provide a graphical characterization of this subspace which can be computed in polynomial time. We also propose a more efficient computation technique based on the Dulmage–Mendelsohn decomposition, but which (up to now) works only in particular cases. Incidentally, the notion of fixed controllable subspace gives the possibility to define an associated fixed control centrality notion which may be of interest in network theory.

The outline of this paper is as follows. In Section 2 structured systems are recalled, together with some known results on structural controllability. Also some elementary observations on controllable subspaces are given, leading to the notion of fixed controllable subspace. In Section 3 a graph theory characterization of the fixed controllable subspace is derived. The results are illustrated by means of some examples in Section 4. The section also contains some remarks on the derived characterizations. In Section 5 an alternative characterization is proposed that possibly offers more insight in the way that the fixed controllable subspace can be constructed from unit vectors. Unfortunately, the proposed method cannot (yet) be applied in all cases. The alternative characterization is illustrated in Section 6. In Section 7 the paper is concluded with some remarks and topics for future research.

2. Structured systems and controllability aspects

2.1. Linear structured systems

We consider a linear system with parameterized entries and denoted by Σ_Λ.

$$\Sigma_\Lambda : \dot{x}(t) = A_\Lambda x(t) + B_\Lambda u(t), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ the input vector, and A_Λ and B_Λ are matrices of appropriate dimensions. The system is called a linear structured system if the entries of the composite matrix $J_\Lambda = [A_\Lambda, B_\Lambda]$ are either fixed zeros or independent parameters (not related by algebraic equations). The vector $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)^\top \in \mathbb{R}^k$, with $^\top$ indicating transposition, denotes the vector of independent (or free) parameters of the composite matrix J_Λ.

For linear structured systems one can study generic properties. A property is said to be generic (or structural) if it is true for all values of the parameter vector Λ outside a proper algebraic variety in the parameter space $\mathbb{R}^k$, see [19]. This can then also be expressed by saying that the property is true for *almost all* values of the vector Λ, since a proper algebraic variety is a variety of zero measure.

For a structured matrix M_Λ, the rank of M_Λ for almost any value of Λ, in the previous sense, is called its generic rank and denoted as g-rank M_Λ. Notice that g-rank M_Λ is also the maximal value of rank M_Λ for any value of Λ, for more details see [19].

A directed graph $G(\Sigma_\Lambda) = (Z, W)$ can be associated with the linear structured system Σ_Λ (1):

- the vertex set is $Z = X \cup U$, where X and U are the state and input sets given by $\{x_1, x_2, \ldots, x_n\}$ and $\{u_1, u_2, \ldots, u_m\}$, respectively,
- the edge set is $W = \{(x_i, x_j) | A_{\Lambda ji} \neq 0\} \cup \{(u_i, x_j) | B_{\Lambda ji} \neq 0\}$, where $A_{\Lambda ji}$ (resp. $B_{\Lambda ji}$) denotes the entry (j, i) of the matrix A_Λ (resp. B_Λ).

[1] This is done for ease of terminology. It would have been more precise to call this subspace the fixed part of the controllable subspace.

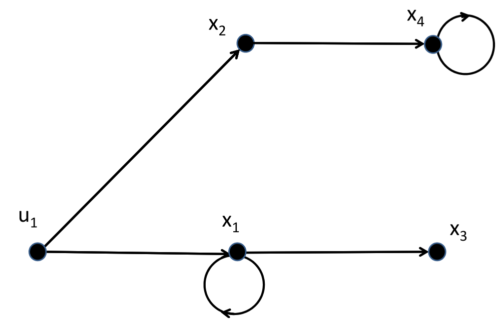

Fig. 1. Graph $G(\Sigma_\Lambda)$ of Example 1.

Recall that a path in $G(\Sigma_\Lambda)$ from a vertex z_{i_0} to a vertex z_{i_q} is a sequence of edges $(z_{i_0}, z_{i_1}), (z_{i_1}, z_{i_2}), \ldots, (z_{i_{q-1}}, z_{i_q})$, such that $z_{i_t} \in Z$ for $t = 0, 1, \ldots, q$ and $(z_{i_{t-1}}, z_{i_t}) \in W$ for $t = 1, 2, \ldots, q$. If $z_{i_0} \in U$ and, $z_{i_q} \in X$, the path is called an input-state path. A path for which $z_{i_0} = z_{i_q}$ is called a circuit. A stem is an input-state path which does not meet the same vertex twice. A system is said to be input-connected if any state vertex is the end vertex of a stem. A cycle is a circuit which does not meet the same vertex twice, except for the initial/end vertex. Two paths are disjoint when they cover disjoint sets of vertices. When some stems and cycles are mutually disjoint, they constitute a set of disjoint stems and cycles.

Example 1. Consider the structured system Σ_Λ with four states and one input, whose parameterized matrices A_Λ and B_Λ are defined as follows.

$$A_\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & \lambda_4 \end{pmatrix}, \quad B_\Lambda = \begin{pmatrix} \lambda_5 \\ \lambda_6 \\ 0 \\ 0 \end{pmatrix}. \quad (2)$$

The corresponding graph is given in Fig. 1. The path (u_1, x_1, x_3) with the circuit on x_4 constitutes a set of disjoint stems and cycles for $G(\Sigma_\Lambda)$.

The notion of structural controllability was introduced and studied by Lin who proved a necessary and sufficient condition for structural controllability in terms of graph-theoretic objects called cacti, see [12]. The following result can be proved to be equivalent to Lin's result, see for instance [13,14].

Theorem 1. *Let Σ_Λ be the linear structured system defined by (1) with associated graph $G(\Sigma_\Lambda)$. The system is structurally controllable if and only if*

- *the system Σ_Λ is input-connected,*
- *g-rank $[A_\Lambda, B_\Lambda] = n$.*

In Example 1, the graph of Σ_Λ is clearly input connected and the first controllability condition is satisfied. The second condition is a little less obvious, but it can be noticed that in

$$[A_\Lambda, B_\Lambda] = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & \lambda_5 \\ 0 & 0 & 0 & 0 & \lambda_6 \\ \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & \lambda_4 & 0 \end{pmatrix}, \quad (3)$$

column 3 being null and columns 2 and 4 being dependent, the generic rank of $[A_\Lambda, B_\Lambda]$ cannot be more than 3. Taking, for example, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 1$ gives a rank 3 for $[A_\Lambda, B_\Lambda]$, therefore the generic rank of $[A_\Lambda, B_\Lambda]$ is 3 and the system is not