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Bispectral analysis for measuring energy-orientation tradeoffs in the control of linear systems

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ABSTRACT

Many characterizations of linear system controllability revolve around the eigenvalue spectrum of the controllability gramian, which is a function of the network dynamics. The gramian spectrum describes the minimum energies associated with inducing movement along orthogonal directions in $\mathbb{R}^n$. Here, we derive an enhanced interpretation of the spectral properties of the gramian in non-minimum energy regimes. Indeed, in a non-minimum energetic regime, an 'excess' of energy is available to the system for at least $(n-1)$ orthogonal state transfers. We show that the utility of this excess energy can be quantified in terms of input orientation, or, simply, the angle between two competing inputs. Based on this notion, we derive the gramian bispectrum, which describes the relationship between energy and orientation among pairs of orthogonal state transfers. The bispectrum reflects a fundamental tradeoff between the energetic and orientation costs in the control of a linear system. We show how this bispectral analysis can provide control characterizations that are not apparent from inspection of the gramian spectrum alone.

1. Introduction

1.1. Motivation

A long-standing topic in control theory, with recent applications in network control [1–3], involves the development of analyses to quantify the controllability of linear systems [4–6]. Many of the approaches that have been developed to address this issue involve study of the spectral properties of the controllability gramian, which, for a controllable linear time-invariant system in the typical form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1}$$

is defined as

$$\mathbf{W}(T) = \int_0^T e^{\mathbf{A}(T-\tau)}\mathbf{B}(\tau)\mathbf{B}'(\tau)e^{\mathbf{A}'(T-\tau)}d\tau. \tag{2}$$

Each eigenvalue of induce motion in the direction of its associated eigenvector. Thus, summary metrics that describe the energetic costs of controlling a linear system can be derived from the gramian spectrum. Such metrics include the trace of the gramian [7,8], the log of its determinant [7], and the maximum eigenvalue of the inverse gramian [7,8]. The latter, in particular, is the 'worst-case' minimum energy required to reach the unit hypersphere at a prescribed time T.

In this paper we seek to characterize the controllability of a system *assuming* that we have at least this minimum energy available to meet control objectives. That is, we assume that the entire unit hypersphere is reachable at time T. What then, can be said about the system under consideration? For a non-uniform gramian spectrum, there will be a gradation of energetic costs, so that we may steer the system in other directions (than the 'worst-case' direction) with less (or much less) energy expenditure. To what use, then, is the excess energy which we have available? Is there a measure of control flexibility that can be realized under this 'surplus energy' scenario?

To answer these questions we consider, in addition to energy, the relative orientation between two inputs $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$, which transfer the state of the system to two different endpoints at time T. Assuming these inputs possess average energy $\gamma_{\mathbf{u}_1}$ and $\gamma_{\mathbf{u}_2}$, then over the time horizon $[0, T]$ the relative orientation, or, mathematically, the average cosine of the angle between $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$, is:

$$\frac{1}{T\sqrt{\gamma_{\mathbf{u}_1}\gamma_{\mathbf{u}_2}}}\int_0^T \mathbf{u}_1'(\tau)\mathbf{u}_2(\tau)d\tau. \tag{3}$$

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If $\gamma_{\mathbf{u}_1}$ and $\gamma_{\mathbf{u}_2}$ are minimal for the transfers in question, then (3) is fixed since $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ can assume only one form. Given excess energy, inputs achieving the transfers may assume a range of possible relative orientations. We propose that this range constitutes a measure of the aforementioned control flexibility, and we provide an analytical development that makes this notion concrete. Specifically, in this paper we make the following contributions:

1. As a function of available energy, we derive the minimum relative orientation (or, average angle) between two inputs, each inducing a transfer to points along a different eigenvector of $\mathbf{W}^{-1}$. A small relative orientation implies that the inputs are relatively flexible in their orientation range. Indeed, we show that as energy tends to infinity the angular difference tends to zero, so that the inputs may become arbitrarily similar in their geometry. The derivation leverages our previous optimal control results in [9] and [10].
2. We derive a second-order spectrum, termed the gramian *bispectrum* that exactly quantifies the tradeoff between energy and orientation, in the above sense, for each pair of eigenvalues of the gramian.
3. We demonstrate how the proposed bispectrum may be used to compare the relative utility of energy in linear systems. We provide examples of such comparisons that show how a bispectral analysis may reveal control properties not apparent from the spectrum alone.

2. Problem formulation

2.1. Geometric interpretation of the controllability gramian

Our subsequent development will pivot on the spectrum Λ of the controllability gramian inverse $\mathbf{W}^{-1}$ (see (2)), defined as the collection of its eigenvalues with ordered labeling, i.e.,

$$\Lambda = \{\lambda_1, \dots, \lambda_n\}, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \tag{4}$$

and corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$.

Since the minimum-energy cost of driving the system from the origin at time $t = 0$ to final point $\mathbf{x}_f$ at time $t = T$ is given by $c(\mathbf{x}_f) = \int_0^T \mathbf{u}'(\tau)\mathbf{u}'(\tau)\mathrm{d}\tau = \mathbf{x}_f'\mathbf{W}^{-1}(T)\mathbf{x}_f$, for a fixed energy $c(\mathbf{x}_f) = c^*$ the gramian essentially prescribes a reachable ellipsoid in n-dimensional space, with vertices located at $\lambda_1\mathbf{v}_1, \dots, \lambda_n\mathbf{v}_n$. This ellipsoid we formally define as

$$\Xi \equiv \left\{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}'\mathbf{W}^{-1}(T)\mathbf{x} = c^*\right\}. \tag{5}$$

Thus, the surface of Ξ encodes the maximal distance to which trajectories can attain at time T under inputs with total energy c^* (starting from the origin). A common way to assay the controllability of a linear system is to deduce the minimum energy, $c^{\min}$, so that system is guaranteed access to a unit-radius hypersphere in $\mathbb{R}^n$ (i.e., min c^* such that $\Xi \supset \left\{\mathbf{x} \in \mathbb{R}^n | \mathbf{x}'\mathbf{x} = 1\right\}$), which as follows from (5), is simply λ_1 in (4).

However, as posed in the Introduction, in this scenario, most of the hypersphere is reachable with an excess of energy (Fig. 1) and we would like to quantify the utility of this excess. Carrying forth the geometric interpretation, it is perhaps intuitive to propose a quantification involving the ratio λ_i/λ_1, $i = 2, \dots, n$, i.e., the eccentricity of the ellipse made by intersecting Ξ with the (v_1, v_i)-plane.

As we will show, this quantity not only results in a useful metric for excess energy utilization, but has a precise interpretation in terms of the relative orientation attainable by putative inputs to the system at hand. We first make concrete the notion of orientation range.

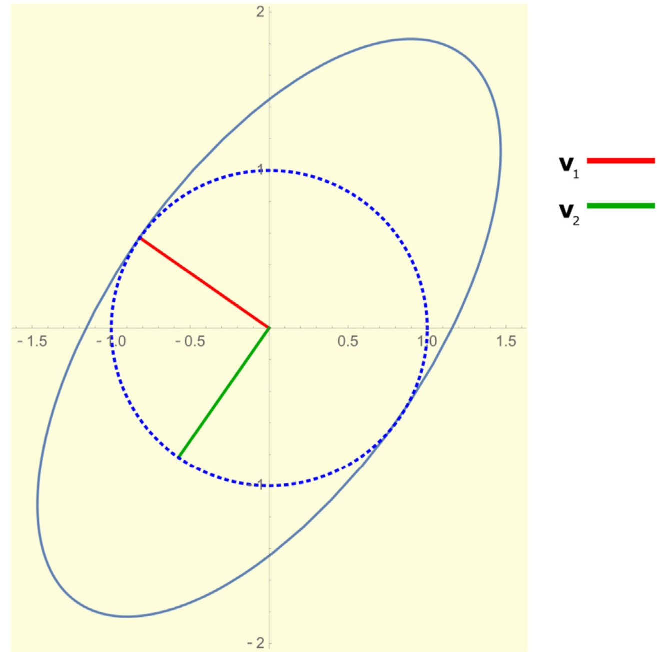

Fig. 1. Unit circle with ellipse prescribed by $\mathbf{W}(T)^{-1}$ of a stable two-dimensional system with the two eigenvectors of $\mathbf{W}(T)^{-1}$ Since the ellipse represents the reachable set with fixed energy λ_1, the distance between the ellipse and circle, in the direction of $\mathbf{v}_2$, encodes the amount of excess energy available if we desire to steer the system to $\mathbf{v}_2$. Note that the figure represents an abstract state space for a 2D linear system, and thus the axes have no explicit units.

2.2. Input orientation and orientation range

We begin by defining formally the expression for relative orientation between two inputs, introduced in (3). For our purposes we will assume one of the inputs is known a priori.

Definition 1 (*Input Orientation*)**.** For reference input $\mathbf{u}_1(t)$ and 'free' input $\mathbf{u}_2(t)$ guiding a system of the form (1) over the time interval $[0, T]$, we define the relative orientation $\mathbf{d}_{\mathbb{J}}(\mathbf{u}_1(t), \mathbf{u}_2(t), t \in [0, T])$–for ease of notation simplified to $\mathbf{d}_{\mathbb{J}}(\mathbf{u}_1, \mathbf{u}_2)$–as follows:

$$\mathbf{d}_{\mathbb{J}}(\mathbf{u}_1, \mathbf{u}_2) = \frac{1}{T\sqrt{\gamma_{\mathbf{u}_1}\gamma_{\mathbf{u}_2}}} \int_0^T \mathbf{u}_1'(\tau)\mathbf{u}_2(\tau)\mathrm{d}\tau, \tag{6}$$

where

$$\gamma_{\mathbf{u}_1} = \frac{1}{T}\int_0^T \mathbf{u}_1'(\tau)\mathbf{u}_1(\tau)\mathrm{d}\tau, \quad \gamma_{\mathbf{u}_2} = \frac{1}{T}\int_0^T \mathbf{u}_2'(\tau)\mathbf{u}_2(\tau)\mathrm{d}\tau \tag{7}$$

are the average energies (up to time T) of $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$, respectively.

We again note that, although we refer to $\mathbf{d}_{\mathbb{J}}(\mathbf{u}_1, \mathbf{u}_2)$ as the relative orientation between inputs $\mathbf{u}_1$ and $\mathbf{u}_2$, in a strict mathematical sense it quantifies the average cosine of the angle between the two inputs.

Under our assumption, the reference is known (thus $\gamma_{\mathbf{u}_1}$ is known), and we seek, given some chosen available energy $\gamma_{\mathbf{u}_2}$, to quantify our freedom in allowing $\mathbf{u}_2(t)$ to assume different geometries while still accomplishing a desired state transfer. We now propose a notion of orientation range based on the maximum similarity between two inputs of fixed energy:

Definition 2 (*Maximum Similarity*)**.** For inputs of energy $\gamma_{\mathbf{u}_1}$, $\gamma_{\mathbf{u}_2}$, the maximum similarity is defined as