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A uniform invariance principle for periodic systems with applications to synchronization

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ABSTRACT

In this paper, we propose an extension of the invariance principle, which is uniform with respect to parameter uncertainties, for the class of periodic ordinary differential equations. This extension allows the derivative of the auxiliary function V, commonly called a Lyapunov function, to be positive in some bounded sets. This important feature has the potential to simplify the problem of exhibiting a function of Lyapunov-type and allows the application of the principle in systems that cannot be treated with the conventional principle, either due to the nonexistence of a Lyapunov-type function or due to the difficulty in exhibiting it. The extension of the invariance principle is useful to obtain estimates of attractors and regions of attraction that are uniform with respect to parameters. The study of synchronization of periodic coupled systems illustrates an application of the principle.

1. Introduction

The invariance principle studies the asymptotic behavior of solutions of differential equations of a system without the need of explicitly calculating them [1–4]. For this purpose, an auxiliary scalar function, called Lyapunov function, is used. Despite its importance in many applications, the invariance principle has some limitations. The main one is the nonexistence of a systematic method for finding the auxiliary scalar function or Lyapunov function. One of the most restrictive conditions in the search for this function is that its derivative has to be negative semi-definite along system trajectories. In several systems, such as chaotic systems with a degree of complexity in their trajectories, it is difficult to find a scalar function satisfying the requirements of the invariance principle and in particular satisfying the condition of the derivative to be negative semi-definite.

A more general version of the invariance principle, called the extension of the invariance principle, simplifies in part this problem by allowing the derivative of the scalar function to assume positive values in some bounded regions of the state space. This extension has been proven for the continuous case [5,6], for the discrete case [7,8], for a class of differential equations with bounded delay [9] and for a class of switched systems [10].

Besides its importance in the theory of stability of nonlinear dynamical systems, the extension of the invariance principle has been successfully applied to stability analysis of electric power systems [11] and problems of synchronization [12,5,8], which was the major motivation of this work.

The extension of the invariance principle does not require the derivative of the scalar function to be negative semi-definite, with a potential to simplify the search for this function. Consequently, many problems that could not be treated by this theory, can now be resolved by the extension of the invariance principle.

This work aims to investigate the existence of extensions of the invariance principle for another class of differential equations, the class of periodic ordinary differential equations.

An invariance principle for periodic nonautonomous differential equations was proved by LaSalle [13] and nearly periodic by Miller [14]. In this paper, an extension of LaSalle's result is proven for the class of periodic systems. This extension follows the same description of the extension proven in [5,6], relaxing the conditions on the derivative of the auxiliary function V, which now can assume positive values in some bounded regions of the state space.

All results of this paper for periodic systems equally apply to autonomous systems, which are periodic systems with an arbitrary period. Consequently, the extensions of the invariance principle proven in [5,6] are particular cases of the ones proven in this paper.

This paper is organized as follows. In Section 2, some properties of periodic dynamical systems are presented and the classic version of the invariance principle for periodic systems is reviewed.

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Finally, the extension of this principle and its uniform version are developed in Sections 3 and 4, respectively. Examples illustrate the application of the developed extension in Sections 3 and 4. An application of the results developed in this paper to the problem of synchronization of coupled periodic systems is explored in Section 5.

2. Preliminaries and the invariance principle for periodic systems

Consider the nonlinear nonautonomous dynamic system

$$\dot{x} = f(t, x) \quad (1)$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ e $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-function. The solution of (1) starting in x_0 at time t_0 will be denoted $s(t, t_0, x_0)$. Solutions of (1) satisfy the following properties of a flow: $s(t_0, t_0, x_0) = x_0, \forall x_0 \in \mathbb{R}^n$ and $s(t, t_1, s(t_1, t_0, x_0)) = s(t, t_0, x_0), \ \forall t, t_1, t_0 \in \mathbb{R}, \ \forall x_0 \in \mathbb{R}^n$.

Definition 2.1. System (1) is periodic with period $T \in \mathbb{R}$ if $f(t + T, x) = f(t, x), \ \forall t \in \mathbb{R}, \ \forall x \in \mathbb{R}^n$.

For a periodic system with period T, solutions have the following additional property: $s(t + kT, t_0 + kT, x_0) = s(t, t_0, x_0)$, $\forall t, t_0 \in \mathbb{R}, \forall x_0 \in \mathbb{R}^n, \forall k \in \mathbb{Z}$ [15].

Definition 2.2. Let $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$. A point $p \in \mathbb{R}^n$ is called a **limit point** of the trajectory $s(t, t_0, x_0)$ if there is a sequence $\{t_i\}$ of numbers in $[t_0, +\infty)$ such that $t_i \to +\infty$ and $\lim_{i\to\infty} \|p - s(t_i, t_0, x_0)\| = 0$. The set of limit points of $s(\cdot, t_0, x_0)$ is called **limit set** of $s(\cdot, t_0, x_0)$ and is denoted $\Omega(t_0, x_0)$.

The property of being closed is true for all limit sets. Furthermore, when the solution is bounded, the limit set is nonempty, bounded and $d(s(t, t_0, x_0), \Omega(t_0, x_0)) \to 0$ when $t \to +\infty$ [15].

Next we will discuss some properties of invariance of limit sets of periodic systems.

Definition 2.3. A set $M \subseteq \mathbb{R}^n$ is called an **invariant set** with respect to the differential equation (1) if, for each $x_0 \in M$, there exists $t_0 \in \mathbb{R}$ such that $s(t, t_0, x_0) \in M, \forall t \in (\omega_-, \omega_+)$, where (ω_-, ω_+) is the maximal interval of existence of solution $s(t, t_0, x_0)$.

Definition 2.4. A set $M \subseteq \mathbb{R}^n$ is called a **positively invariant set** with respect to the differential equation (1) if, for each $x_0 \in M$, there exists $t_0 \in \mathbb{R}$ such that $s(t, t_0, x_0) \in M, \forall t \in [t_0, \omega_+)$.

In general, limits sets of nonautonomous systems are not invariant. In the particular case of periodic and autonomous systems, limits sets are also invariant.

Lemma 2.1 ([15]). *Let $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$. Let $s(\cdot, t_0, x_0)$ be the solution of system (1) starting in $x_0 \in \mathbb{R}^n$ at time $t_0 \in \mathbb{R}$ and assume s is defined for all $t \in [t_0, \infty)$. Suppose that system (1) is periodic, then $\Omega(t_0, x_0)$ is an invariant set.*

The proof of Lemma 2.1 explores the periodicity of the vector field to prove invariance. Although this proof is already known in literature [15], it will be presented here because it contains ideas that will be strongly explored in the proofs of the results developed in this paper.

Proof of Lemma 2.1. Let T be the period of (1) and let $p \in \Omega(t_0, x_0)$. We will show the existence of an initial time $\tau \in \mathbb{R}_+$ such that $s(t, \tau, p) \in \Omega(t_0, x_0), \forall \ t \in (\omega_-, \omega_+)$.

Take $p \in \Omega(t_0, x_0)$. Then there is a sequence $\{t_i\}$, with $t_i \to \infty$ as $i \to \infty$, such that $\lim_{i\to\infty} \|p - s(t_i, t_0, x_0)\| = 0$.

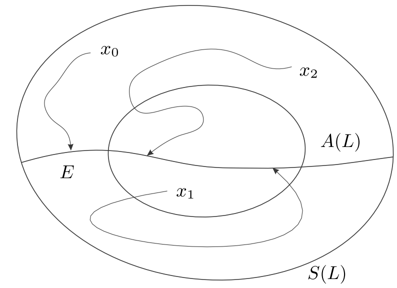

Fig. 1. Geometric interpretation of the invariance principle for periodic systems.

For each i, find $k_i \in \mathbb{Z}$ such that $t_i - k_iT \in [0, T)$. Then the sequence $\{\tau_i\} = \{t_i - k_iT\}$ is bounded and therefore admits a convergent subsequence. Choose such a subsequence $\{\tau_i'\} = \{t_i' - k_i'T\}$ and rewrite it as $\{\tau_i\}$. Let $\tau \in [0, T]$ be the limit of this subsequence. Solutions depend continuously on the initial conditions and on the time, then $s(t, \tau, p) = \lim_{i\to\infty} s[t, \tau, s(t_i, t_0, x_0)] = \lim_{i\to\infty} s[t + k_iT, \tau + k_iT, s(t_i, t_0, x_0)] = \lim_{i\to\infty} s[t + k_iT, t_i, s(t_i, t_0, x_0)] = \lim_{i\to\infty} s[t + k_iT, t_0, x_0]$, where we use the fact that $\tau = \lim_{i\to\infty}(t_i - k_iT)$. Therefore $s(t, \tau, p) \in \Omega(t_0, x_0)$. □

For a scalar C^1-function $V : \mathbb{R}^n \to \mathbb{R}$, two level sets will be considered:

$$S(L) := \{x \in \mathbb{R}^n : \exists t \in \mathbb{R} \text{ such that } V(t, x) < L\}$$

and

$$A(L) := \{x \in \mathbb{R}^n : V(t, x) < L, \forall t \in \mathbb{R}\}.$$

Clearly $A(L) \subseteq S(L)$. These sets will be important for the development of the invariance principle and its extensions.

Under the assumption that $\dot{V} \leq 0$, for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, it can be easily proven that $S(L)$ is a positively invariant set. Level set $A(L)$ is not necessarily positively invariant, but it also can be proven that solutions starting in $A(L)$ do not leave $S(L)$. Exploring these invariance properties of level sets, the invariance principle for periodic systems [15] is stated as follows:

Theorem 2.1 ([15] *Invariance Principle for Periodic Systems*). *Suppose that system (1) is periodic and $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a periodic C^1-function with the same period of system (1). Let $L \in \mathbb{R}$ be a real constant, and consider the sets $S(L)$ and $A(L)$. Suppose that $S(L)$ is bounded and $\dot{V}(t, x) \leq 0, \forall t \in \mathbb{R}, \forall x \in S(L)$. Define $E := \{x \in \overline{S(L)} : \exists t \in \mathbb{R}$ such that $\dot{V}(t, x) = 0\}$ and let B be the largest invariant set contained in E. Then*

(i) *$x_0 \in A(L) \Rightarrow s(t, t_0, x_0) \to B$ as $t \to +\infty$ for all $t_0 \in \mathbb{R}$;*
(ii) *$x_0 \in S(L) \Rightarrow \exists t_0$ such that $s(t, t_0, x_0) \to B$ as $t \to +\infty$.*

Fig. 1 illustrates Theorem 2.1. For the initial condition $x_1 \in A(L)$ in Fig. 1, the solution does not leave $S(L)$ and tends to the largest invariant set in E for any t_0. For the initial conditions at x_0 and x_2 in $S(L)$, there are t_0 and t_2 such that the solutions starting in x_0 and x_2 at these times, respectively, do not leave $S(L)$ and tend to the largest invariant set B in E.

3. Extension of the invariance principle for periodic systems

In this section, an extension of the invariance principle for periodic systems is developed. The key feature of this extension is the possibility of the derivative of the auxiliary scalar function V to assume positive values in bounded regions of the state space.

When the derivative of V is not negative semi-definite, we can still achieve similar properties of invariance of level sets $S(L)$ and $A(L)$ imposing some control over the regions where the derivative of V is positive. Lemmas 3.1 and 3.2 study these properties.