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Lyapunov and converse Lyapunov theorems for stochastic semistability☆

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ABSTRACT

This paper develops Lyapunov and converse Lyapunov theorems for stochastic semistable nonlinear dynamical systems. Semistability is the property whereby the solutions of a stochastic dynamical system almost surely converge to (not necessarily isolated) Lyapunov stable in probability equilibrium points determined by the system initial conditions. Specifically, we provide necessary and sufficient Lyapunov conditions for stochastic semistability and show that stochastic semistability implies the existence of a continuous Lyapunov function whose infinitesimal generator decreases along the dynamical system trajectories and is such that the Lyapunov function satisfies inequalities involving the average distance to the set of equilibria.

1. Introduction

The aim of this paper is to develop Lyapunov and converse Lyapunov theorems for stochastic semistability. Semistability is the property of a dynamical system whereby its trajectories converge to (not necessarily isolated) Lyapunov stable equilibria. Semistability, rather than asymptotic stability, is the appropriate notion of stability for systems having a continuum of equilibria. Examples of such systems arise in chemical kinetics [1], adaptive control [2], compartmental modeling [3], thermodynamics [4] and, more recently, collaborative control of a network of autonomous agents [5,6]. In all these examples, the system trajectories converge to limit points that depend continuously on the system initial conditions.

It is important to note that semistability is not merely equivalent to asymptotic stability of the set of equilibria. Indeed, it is possible for a trajectory to converge to the set of equilibria without converging to any one equilibrium point as examples in [2] show. Conversely, semistability does not imply that the equilibrium set is asymptotically stable in any accepted sense. This is because stability of sets is defined in terms of distance (especially in case of noncompact sets), and it is possible to construct examples in which the system is semistable, but the domain of semistability contains no ε-neighborhood (defined in terms of the distance) of the (noncompact) equilibrium set, thus ruling out asymptotic stability of the equilibrium set. Hence, semistability and set stability of the equilibrium set are independent notions.

For linear deterministic systems, semistability was originally defined in [7] and applied to matrix second-order systems in [8]. Refs. [2,9] extended the notion of semistability to nonlinear deterministic systems and gave Lyapunov results for semistability. Semistability was also addressed in [5,6] for consensus protocols in nonlinear dynamical networks, with [6] giving new Lyapunov theorems as well as the first converse Lyapunov theorem for semistability which holds with a smooth (i.e., infinitely differentiable) Lyapunov function.

In numerous applications where dynamical models are used to describe the behavior of natural and engineering systems, stochastic components and random disturbances are incorporated into the models. The stochastic aspects of the models are used to quantify system uncertainty as well as the dynamic relationships of sequences of random events between system–environment interactions. For example, stochastic modeling can be used to capture communication uncertainty between agents in a network, wherein the evolution of each link of the random network communication topology follows a Markov process. In this case, the development of almost sure consensus of multiagent systems with nonlinear stochastic dynamics under distributed nonlinear consensus protocols is necessary. And from a practical viewpoint, it is not sufficient to only guarantee that the network almost surely converges to a state of consensus since steady-state convergence

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is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from a state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable in probability, and consequently, stochastically semistable.

Using a notion of stochastic semistability, almost sure consensus of multiagent systems under distributed nonlinear protocols over random networks is addressed in [10]. The authors in [10] consider stochastic systems driven by a discrete-valued, right-continuous strong Markov excitation process. In this paper, we extend the notion of semistability to nonlinear stochastic systems involving Markov diffusion processes that have a continuum of equilibrium solutions. In particular, we develop almost sure convergence and stochastic Lyapunov stability properties to address almost sure semistability requiring the trajectories of a nonlinear stochastic dynamical system to converge almost surely to a set of equilibrium solutions, wherein every equilibrium solution in the set is almost surely Lyapunov stable. Furthermore, we provide necessary and sufficient Lyapunov conditions for semistability and show that semistability implies the existence of a continuous Lyapunov function whose infinitesimal generator decreases along the dynamical system trajectories and is such that the Lyapunov function satisfies inequalities involving the average distance to the set of equilibria.

2. Notation, definitions, and mathematical preliminaries

In this section, we establish notation, definitions, and develop mathematical preliminaries necessary for developing the results in this paper. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ denotes the set of positive real numbers, $\overline{\mathbb{R}}_+$ denotes the set of nonnegative numbers, and $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors. We write $\mathcal{B}_\varepsilon(x)$ for the *open ball centered* at x with *radius* ε, $\|\cdot\|$ for the Euclidean vector norm, $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm, A^{T} for the transpose of the matrix A, and I_n or I for the $n \times n$ identity matrix. Furthermore, $\mathfrak{B}^n$ denotes the σ-algebra of Borel sets in $\mathcal{D} \subseteq \mathbb{R}^n$ and $\mathfrak{S}$ denotes a σ-algebra generated on a set $\mathcal{S} \subseteq \mathbb{R}^n$.

We model a stochastic dynamical system $\mathcal{G}$ generating a stochastic process $x : \overline{\mathbb{R}}_+ \times \Omega \to \mathcal{D}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω denotes the sample space, $\mathcal{F}$ denotes a σ-algebra of subsets of Ω, and $\mathbb{P}$ defines a probability measure on the σ-algebra $\mathcal{F}$; that is, $\mathbb{P}$ is a nonnegative countably additive set function on $\mathcal{F}$ such that $\mathbb{P}(\Omega) = 1$ [11]. We equip the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a continuous-time filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by a standard d-dimensional Wiener process $w(t)$ up to time t inclusively and satisfying $\mathcal{F}_\tau \subset \mathcal{F}_t$, $0 \leq \tau < t$, such that $\{\omega \in \Omega : x(t,\omega) \in \mathcal{B}\} \in \mathcal{F}_t$, $t \geq 0$, for all Borel sets $\mathcal{B} \subset \mathbb{R}^n$ contained in the Borel σ-algebra $\mathfrak{B}^n$. Here we use the notation $x(t)$ to represent the stochastic process $x(t,\omega)$ omitting its dependence on ω.

We denote the set of equivalence classes of measurable, integrable, and square-integrable $\mathbb{R}^n$ or $\mathbb{R}^{n\times m}$ (depending on context) valued random processes on $(\Omega, \mathcal{F}, \mathbb{P})$ over the semi-infinite parameter space $[0,\infty)$ by $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, respectively, where the equivalence relation is the one induced by $\mathbb{P}$-almost-sure equality. In particular, elements of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ take finite values $\mathbb{P}$-almost surely (a.s.). Hence, depending on the context, $\mathbb{R}^n$ will denote either the set of $n \times 1$ real variables or the subspace of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ comprising of $\mathbb{R}^n$ random processes that are constant almost surely. All inequalities and equalities involving random processes on $(\Omega, \mathcal{F}, \mathbb{P})$ are to be understood to hold $\mathbb{P}$-almost surely.

Given $x \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$, $\{x = 0\}$ denotes the set $\{\omega \in \Omega : x(t,\omega) = 0\}$, and so on. Given $x \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{E} \in \mathcal{F}$, we say x is nonzero on $\mathcal{E}$ if $\mathbb{P}(\{x = 0\} \cap \mathcal{E}) = 0$. Furthermore, given $x \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and a σ-algebra $\mathcal{E} \subseteq \mathcal{F}$, $\mathbb{E}^{\mathbb{P}}[x]$ and $\mathbb{E}^{\mathbb{P}}[x|\mathcal{E}]$ denote, respectively, the expectation of the random variable x and the conditional expectation of x given $\mathcal{E}$, with all moments taken under the measure $\mathbb{P}$. Here, for simplicity of exposition, we omit the symbol $\mathbb{P}$ in denoting expectation, and similarly for conditional expectation. Specifically, we denote the expectation with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by $\mathbb{E}[\ \cdot\ ]$, and similarly for conditional expectation.

Finally, we write $\mathrm{tr}(\cdot)$ for the trace operator, $(\cdot)^{-1}$ for the inverse operator, $V'(x) \triangleq \frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x, $V''(x) \triangleq \frac{\partial^2 V(x)}{\partial x^2}$ for the Hessian of V at x, and $\mathcal{H}_n$ for the Hilbert space of random vectors $x \in \mathbb{R}^n$ with finite average power, that is, $\mathcal{H}_n \triangleq \{x : \Omega \to \mathbb{R}^n : \mathbb{E}[x^{\mathrm{T}}x] < \infty\}$. For an open set $\mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{H}_n^{\mathcal{D}} \triangleq \{x \in \mathcal{H}_n : x : \Omega \to \mathcal{D}\}$ denotes the set of all the random vectors in $\mathcal{H}_n$ induced by $\mathcal{D}$. Similarly, for every $x_0 \in \mathbb{R}^n$, $\mathcal{H}_n^{x_0} \triangleq \{x \in \mathcal{H}_n : x \overset{\text{a.s.}}{=} x_0\}$. Furthermore, C^2 denotes the space of real-valued functions $V : \mathcal{D} \to \mathbb{R}$ that are two-times continuously differentiable with respect to $x \in \mathcal{D} \subseteq \mathbb{R}^n$.

Definition 2.1 (*[12]*)**.** Let $(S, \mathfrak{S})$ and $(T, \mathfrak{T})$ be measurable spaces, and let $\mu : S \times \mathfrak{T} \to \overline{\mathbb{R}}_+$. If the function $\mu(s, B)$ is $\mathfrak{S}$-measurable in $s \in S$ for a fixed $B \in \mathfrak{T}$ and $\mu(s, B)$ is a probability measure in $B \in \mathfrak{T}$ for a fixed $s \in S$, then μ is called a *(probability) kernel* from S to T. Furthermore, for $s \leq t$, the function $\mu_{s,t} : S \times \mathfrak{S} \to \mathbb{R}$ is called a *regular conditional probability measure* if $\mu_{s,t}(\cdot, \mathfrak{S})$ is measurable, $\mu_{s,t}(S, \cdot)$ is a probability measure, and $\mu_{s,t}(\cdot,\cdot)$ satisfies

$$\begin{aligned}\mu_{s,t}(x(s), B) &= \mathbb{P}(x(t) \in B|x(s)) \\ &= \mathbb{P}(x(t) \in B|\mathcal{F}_s) \quad \text{a.s.}, \ x(\cdot) \in \mathcal{H}_n. \end{aligned} \tag{1}$$

Any family of regular conditional probability measures $\{\mu_{s,t}\}_{s\leq t}$ satisfying the Chapman–Kolmogorov equation [11] is called a *semigroup of Markov kernels*. The Markov kernels are called *time homogeneous* if and only if $\mu_{s,t} = \mu_{0,t-s}$ holds for all $s \leq t$.

Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by

$$\mathrm{d}x(t) = f(x(t))\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \ t \in \mathcal{I}_{x(0)}, \tag{2}$$

where, for every $t \in \mathcal{I}_{x_0}$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$ is a $\mathcal{F}_t$-measurable random state vector, $x(0) \in \mathcal{H}_n^{x_0}$, $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{D}$, $w(t)$ is a d-dimensional independent standard Wiener process (i.e., Brownian motion) defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, $x(0)$ is independent of $(w(t) - w(0))$, $t \geq 0$, $f : \mathcal{D} \to \mathbb{R}^n$ and $D : \mathcal{D} \to \mathbb{R}^{n\times d}$ are continuous, $\mathcal{E} \triangleq f^{-1}(0) \cap D^{-1}(0) \triangleq \{x \in \mathcal{D} : f(x) = 0 \text{ and } D(x) = 0\}$ is nonempty, and $\mathcal{I}_{x(0)} = [0, \tau_{x(0)})$, $0 \leq \tau_{x(0)} \leq \infty$, is the maximal interval of existence for the solution $x(\cdot)$ of (2). An *equilibrium point* of (2) is a point $x_{\mathrm{e}} \in \mathbb{R}^n$ such that $f(x_{\mathrm{e}}) = 0$ and $D(x_{\mathrm{e}}) = 0$. It is easy to see that x_{e} is an equilibrium point of (2) if and only if the constant stochastic process $x(\cdot) \overset{\text{a.s.}}{=} x_{\mathrm{e}}$ is a solution of (2). We denote the set of equilibrium points of (2) by $\mathcal{E} \triangleq \{\omega \in \Omega : x(t,\omega) = x_{\mathrm{e}}\} = \{x_{\mathrm{e}} \in \mathcal{D} : f(x_{\mathrm{e}}) = 0 \text{ and } D(x_{\mathrm{e}}) = 0\}$.

The filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is clearly a real vector space with addition and scalar multiplication defined componentwise and pointwise. A $\mathbb{R}^n$-valued stochastic process $x : [0,\tau] \times \Omega \to \mathcal{D}$ is said to be a *solution* of (2) on the time interval $[0,\tau]$ with initial condition $x(0) \overset{\text{a.s.}}{=} x_0$ if $x(\cdot)$ is *progressively measurable* (i.e., $x(\cdot)$ is nonanticipating and measurable in t and ω) with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, $D \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, and

$$\begin{aligned} x(t) &= x_0 + \int_0^t f(x(\sigma))\mathrm{d}\sigma \\ &\quad + \int_0^t D(x(\sigma))\mathrm{d}w(\sigma) \quad \text{a.s.}, \ t \in [0,\tau], \end{aligned} \tag{3}$$

where the integrals in (3) are Itô integrals.