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Data-driven interpolation of dynamical systems with delay

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ABSTRACT

We present a data-driven realization for systems with delay, which generalizes the Loewner framework. The realization is obtained with low computational cost directly from measured data of the transfer function. The internal delay is estimated by solving a least-square optimization over some sample data. Our approach is validated by several examples, which indicate the need for preserving the delay structure in the reduced model.

1. Introduction

Nowadays, it is common to describe a physical system by a mathematical surrogate model. Such models are often given by (partial) differential equations and may be used for analysis, control, and optimization. The demand for high fidelity models results in large-scale dynamical systems, for which classical numerical methods may be too time or memory consuming. Hence, an analytically justified and numerically stable approximation of the input–output map is desirable leading to the field of model order reduction (MOR) (for an overview see [1–3]). Many of these MOR methods require access to the state-space realization of the full system. This assumption can be relaxed by employing data-driven realization techniques that construct models directly from measurements. These models can then be further reduced if necessary.

Let $H : \mathbb{C} \to \mathbb{C}^{p,m}$ denote the transfer function of a system, where m and p are the numbers of inputs and outputs, respectively. Since the input–output behavior of a system is characterized by its transfer function, measurements of H seem appropriate to construct a realization. We assume measurements $H(\lambda_i)r_i = w_i$ and $\ell_i H(\mu_i) = v_i$ to be given, i.e., we have right and left interpolation data, given by

$$\begin{aligned}&\{(\lambda_i, r_i, w_i) \mid \lambda_i \in \mathbb{C},\ r_i \in \mathbb{C}^m,\ w_i \in \mathbb{C}^p\},\\ &\{(\mu_i, \ell_i, v_i) \mid \mu_i \in \mathbb{C},\ \ell_i^{\mathrm{T}} \in \mathbb{C}^p,\ v_i^{\mathrm{T}} \in \mathbb{C}^m\},\end{aligned} \tag{1}$$

respectively, for $i = 1, \ldots, \rho$. Examples of measurements yielding data in the form (1) are scattering parameters for frequency response objects (S-parameters) and admittance parameters for interconnects, which can be obtained by a vector network analyzer [4].

In this paper we assume that the transfer function is based on a system with (possibly unknown) delay and study a *generalized realization problem with internal delay*: Given the data (1), construct matrices E_ρ, $A_{1,\rho}$, $A_{2,\rho}$, B_ρ, and C_ρ, such that the transfer function

$$H_\rho(s) = C_\rho \left(sE_\rho - A_{1,\rho} - \mathrm{e}^{-\tau s}A_{2,\rho}\right)^{-1} B_\rho \tag{2}$$

with delay $\tau \geq 0$ interpolates the data, i.e.,

$$w_i = H(\lambda_i)r_i = H_\rho(\lambda_i)r_i \quad \text{and} \quad v_i = \ell_i H(\mu_i) = \ell_i H_\rho(\mu_i)$$

for $i = 1, \ldots, \rho$. The transfer function (2) corresponds to a realization $\Sigma_\rho = (E_\rho, A_{1,\rho}, A_{2,\rho}, B_\rho, C_\rho)$ of the form

$$E_\rho \dot{x}_\rho(t) = A_{1,\rho}x_\rho(t) + A_{2,\rho}x_\rho(t-\tau) + B_\rho u(t), \tag{3a}$$
$$y_\rho(t) = C_\rho x_\rho(t), \tag{3b}$$

which serves as a low-dimensional model. Note that we allow the matrix E_ρ to be singular such that also neutral and advanced equations are covered by system (3) [5].

A generalized realization problem without delay for the data (1) is solved in [6], leading to the Loewner framework. The resulting realization is a generalized state-space representation of the form

$$\begin{aligned}-\mathbb{L}\dot{x}(t) &= -\mathbb{L}_\sigma x(t) + Vu(t),\\ y(t) &= Wx(t),\end{aligned} \tag{4}$$

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where $\mathbb{L}$ and $\mathbb{L}_\sigma$ are the Loewner matrix and shifted Loewner matrix, respectively, and V and W are matrices consisting of the data (for more details see Section 2).

The rate of change of realistic models often depends not only on the current time point, but also on the configuration at previous time instances, which leads to time-delay systems. Popular examples are nonlinear optics, chemical reactor systems, and delayed feedback control (cf. [7] and the references within). Finding a realization of a system with delay by means of the Loewner framework results in the system (4), that does not feature the delay term and hence cannot reflect the dynamics of the inherently infinite-dimensional delay system.

In this paper, we propose a generalization of the Loewner realization to systems with delay. The main contributions are described in the following.

- We consider general conditions for interpolating the transfer function (2) in Section 3. For this purpose, ideas from the moment matching literature [8,9] are extended to descriptor systems with delay.
- Based on the interpolation conditions, we derive a framework to obtain a realization (3) with the coupling $A_{2,\rho} = \alpha E_\rho + \beta A_{1,\rho}$ with scalar parameters α and β (see Section 4).
- Since the delay is unknown in general, we propose a methodology to determine a delay value $\tau_{\kappa,\rho}$, which is optimal in the sense that it minimizes the interpolation error for a set of sampled test data of the transfer function (see Section 5). In the same fashion, optimal values for the parameters α and β may be calculated.

In Section 6 we apply the proposed framework to several examples. A comparison to the original Loewner framework reveals the necessity of preserving the delay structure in the realization.

2. Notation and preliminary results

Recall that the realization (3) is given by

$$E_\rho \dot{x}_\rho(t) = A_{1,\rho} x_\rho(t) + A_{2,\rho} x_\rho(t-\tau) + B_\rho u(t), \tag{5a}$$
$$y_\rho(t) = C_\rho x_\rho(t), \tag{5b}$$

where $x_\rho(t) \in \mathbb{R}^r$ for $r \le \rho$, $u(t) \in \mathbb{R}^m$, and $y_\rho(t) \in \mathbb{R}^p$ denote, respectively, the *state*, *input*, and *output* of the model. As common in the delay literature, the right-hand derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ of a piecewise smooth function f is denoted by $\dot{f}$ [10]. The symbol I_n stands for the identity matrix of dimension $n \times n$, e_i is the ith unit vector of suitable dimension, and δ_{ij} is the Kronecker delta. The input u is assumed to be sufficiently smooth and the system (5) is equipped with the initial condition (also called *history function*)

$$x(t) = \phi(t) \quad \text{for } t \in (-\tau, 0], \tag{6}$$

which is assumed to be identically zero, i.e., $\phi \equiv 0$. If $\det(sE_\rho - A_{1,\rho} - \mathrm{e}^{-\tau s} A_{2,\rho})$ is not vanishing identically, then the Laplace transform of (5) yields the transfer function

$$H_\rho(s) = C_\rho \left(sE_\rho - A_{1,\rho} - \mathrm{e}^{-\tau s} A_{2,\rho}\right)^{-1} B_\rho. \tag{7}$$

We briefly recall the Loewner realization introduced in [6]. The Loewner matrix $\mathbb{L}$ and shifted Loewner matrix $\mathbb{L}_\sigma$ are defined componentwise via

$$[\mathbb{L}]_{i,j} = \frac{v_i r_j - \ell_i w_j}{\mu_i - \lambda_j} \quad \text{and} \quad [\mathbb{L}_\sigma]_{i,j} = \frac{\mu_i v_i r_j - \lambda_j \ell_i w_j}{\mu_i - \lambda_j}. \tag{8}$$

For the ease of presentation we introduce the matrices

$$\begin{aligned} \Lambda &= \mathrm{diag}(\lambda_1, \ldots, \lambda_\rho), & M &= \mathrm{diag}(\mu_1, \ldots, \mu_\rho), \\ R &= \begin{bmatrix} r_1 & \cdots & r_\rho \end{bmatrix}, & L^{\mathrm{T}} &= \begin{bmatrix} \ell_1^{\mathrm{T}} & \cdots & \ell_\rho^{\mathrm{T}} \end{bmatrix}, \\ W &= \begin{bmatrix} w_1 & \cdots & w_\rho \end{bmatrix} \quad \text{and} & V^{\mathrm{T}} &= \begin{bmatrix} v_1^{\mathrm{T}} & \cdots & v_\rho^{\mathrm{T}} \end{bmatrix}. \end{aligned}$$

Remark 2.1. Both Loewner matrices can be assembled efficiently via matrix–matrix operations of size $\rho \times \rho$ using standard tools for scalar, vector, and matrix operations (BLAS), see [6] for details. This is important since – unlike in the classical model reduction setting – ρ is the number of available data points and might be large.

Theorem 2.2 (*[6, Lemma 5.1]*). *Let* $\det(\tilde{s}\mathbb{L} - \mathbb{L}_\sigma) \neq 0$ *for all* $\tilde{s} \in \{\lambda_i\} \cup \{\mu_i\}$. *Then the system*

$$\begin{aligned} -\mathbb{L}\dot{x}_\rho(t) &= -\mathbb{L}_\sigma x_\rho(t) + Vu(t), \\ y_\rho(t) &= W x_\rho(t) \end{aligned} \tag{9}$$

is a minimal realization of an interpolant of the data, i.e., its transfer function $H_\rho(s) = W(\mathbb{L}_\sigma - s\mathbb{L})^{-1} V$ *interpolates the data (1).*

Let ε be the machine precision. If $\det(\tilde{s}\mathbb{L} - \mathbb{L}_\sigma) = \mathcal{O}(\varepsilon)$ for some $\tilde{s} \in \{\lambda_i\} \cup \{\mu_i\}$, then one can use the truncated singular value decomposition (SVD) [11] of $s\mathbb{L} - \mathbb{L}_\sigma$ to weaken the regularity condition in Theorem 2.2 (cf. [6]).

Recently, a generalization of the Loewner framework for a special class of delay systems, where $A_{1,\rho} = 0$ in (5), was introduced in [12]. We give the result here in slightly different notation.

Theorem 2.3 (*[12, Theorem 3]*). *Let* $\mathbb{L}^{(\tau)}$ *and* $\mathbb{L}_\sigma^{(\tau)}$ *denote the Loewner matrix and shifted Loewner matrix, respectively, associated with the transformed data*

$$(\lambda_i \mathrm{e}^{\tau\lambda_i}, r_i, \mathrm{e}^{-\tau\lambda_i} w_i) \quad \text{and} \quad (\mu_i \mathrm{e}^{\tau\mu_i}, \ell_i, \mathrm{e}^{-\tau\mu_i} v_i).$$

In particular assume $\lambda_i \mathrm{e}^{\tau\lambda_i} \neq \mu_j \mathrm{e}^{\tau\mu_j}$ *for* $i,j = 1, \ldots, \rho$. *If* $\det(\tilde{s}\mathbb{L}^{(\tau)} - \mathrm{e}^{-\tau\tilde{s}} \mathbb{L}_\sigma^{(\tau)}) \neq 0$ *for all* $\tilde{s} \in \{\lambda_i\} \cup \{\mu_i\}$, *then the transfer function*

$$H_\rho(s) = W \mathrm{e}^{-\tau\Lambda} \left(\mathrm{e}^{-\tau s} \mathbb{L}_\sigma^{(\tau)} - s\mathbb{L}^{(\tau)}\right)^{-1} \mathrm{e}^{-\tau M} V$$

of the system

$$\begin{aligned} -\mathbb{L}^{(\tau)} \dot{x}_\rho(t) &= -\mathbb{L}_\sigma^{(\tau)} x_\rho(t-\tau) + \mathrm{e}^{-\tau M} V u(t), \\ y_\rho(t) &= W \mathrm{e}^{-\tau\Lambda} x_\rho(t) \end{aligned}$$

is an interpolant of the original data (1).

3. Two-sided interpolation and parametrization of all interpolants

Subsequently, we present a general framework for two-sided interpolation of a linear time-invariant time-delay descriptor system of the form

$$\begin{aligned} E\dot{x}(t) &= A_1 x(t) + A_2 x(t-\tau) + Bu(t), \\ y(t) &= Cx(t) \end{aligned} \tag{10}$$

with state-space dimension n, input dimension m, output dimension p, and matrices E, A_1, A_2, B, and C of appropriate size. The transfer function of the system is

$$H(s) = C\left(sE - A_1 - \mathrm{e}^{-\tau s} A_2\right)^{-1} B \tag{11}$$

and we assume that $\det(sE - A_1 - \mathrm{e}^{-\tau s} A_2) \not\equiv 0$. Interpolation aims in constructing a reduced model (3) with $\rho \ll n$ such that H_ρ interpolates H at given frequencies $s \in \mathbb{C}$. Classical (tangential) interpolation [13] as a MOR method was recently extended to a general coprime form [14] that includes (11) as a special case. However the authors give only sufficient conditions for the interpolation. To transfer the results to the delay realization problem we need also the necessary conditions that allow for a parametrization of the reduced model. To this end we generalize ideas from [8,9] to delay descriptor systems of the form (10).