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A positivity-based approach to delay-dependent stability of systems with large time-varying delays ☆

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ABSTRACT

In this paper we propose new explicit tests for positivity and exponential stability of systems with large time-varying delays. Our approach is based on nonoscillation of solutions of the corresponding diagonal scalar delay differential equations. Numerical examples illustrate the efficiency of the results.

1. Introduction

Positive systems appear in various models that are composed of interconnected subsystems, where each subsystem presents a compartment. Compartments exchange variable nonnegative quantities of material with conservation laws describing transfer, accumulation, and outflows between compartments and the environment [1]. Transfers between the compartments have to account time for material, energy, or information in transit between the compartments. This leads to analysis of delay systems of the following form

$$x'(t) + A_0(t)x(t) + \sum_{k=1}^{m} A_k(t)x(t-\theta_k(t)) = 0,$$

$$t \in [0, +\infty), \tag{1.1}$$

$$x(\xi) = \varphi(\xi), \quad \xi < 0, \tag{1.2}$$

where $\varphi : (-\infty, 0) \to \mathbb{R}^n$ is a given continuous n-vector function, defining what can be substituted into the equation for $t - \theta_k(t) < 0, A_k(t) = \{a_{ij}^k(t)\}_{i,j=1,\ldots,n}$, $k = 0, \ldots, m$, are $n \times n$ matrices with bounded piecewise continuous entries, $x(t) = col\,\{x_1(t), \ldots, x_n(t)\} \in \mathbb{R}^n$ is n-vector with absolutely continuous components, the delays $\theta_k(t)$ are measurable bounded functions for $k = 1, \ldots, m$.

In this paper, we deal with the positivity-based stability analysis of (1.1). This approach was started in [2], and was further developed in [3–7]. For difference and delay differential systems this approach was developed in [8–11,6,1,12–18]. For applications of this approach to additive neural networks see [19,13]. In all the above works that treat (1.1) it is assumed that there is a non-delayed term $A_0(t)x(t)$ with positive terms on the main diagonal of A_0. These diagonal terms should be sufficiently large in order to achieve dominance of the main diagonal of the matrix A_0 over all the other terms (see, for example, the condition (5) of Theorem 3.1 in [1] and condition (iii) of Theorem III. I in [16]). Such an assumption can be interpreted as follows: the diagonal ordinary differential equations describing every compartment, should be exponentially stable, and interconnections between different compartments should be sufficiently weak in order not to destabilize the system (1.1).

The approaches of above papers are not applicable to stabilization of an open-loop unstable system

$$x'(t) + A_0(t)x(t) + \sum_{k=1}^{m} A_k(t)x(t-\theta_k(t)) = u(t),$$

$$t \in [0, +\infty), \tag{1.3}$$

by the delayed feedback $u(t) = -\sum_{k=1}^{m} B(t)x(t-\tau_k(t))$, with $\tau_k(t) > \theta_k(t) > 0$ for $t \in [0, +\infty)$, $k = 1, \ldots, m$. The latter inequality may naturally appear in applications. The presence of

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time-delay in the control input may destabilize the closed-loop systems, as pointed out, for example, in [20,21,7]. One of the popular approaches used to cope with delays in the input is the predictor-based approach (see e.g. [22]). Recent developments in this area were presented in [23]. Another way to cope with delays in the input is to reduce systems of the delay differential equations to systems of "integral equations" (see, for example, [24,4] and the references therein). This approach allows to deal with variable delays and coefficients leading to simple stability conditions in a form of inequalities. Based on this approach the positivity-based stability analysis-results were provided in [25–28], where a smallness of delays on the main diagonal was assumed instead of their absence (see, for example, Proposition 2.3). Positivity-based stability of neutral systems with small delays on the main diagonal was considered in [25,29,17] see also the recent paper [30]. Results on stability of systems with distributed delay can be found, for example, in [31,18], where "smallness of delays" on the main diagonal is also assumed.

In the present paper, for the first time, the stability conditions for systems with large time-varying delays are provided under assumption of the closeness of the delays instead of the delays' smallness. Theorems 3.2 and 3.3 present sufficient conditions for the exponential stability in this case. Theorem 3.4 generalizes to systems with large delays the classical theorem about equivalence of the exponential stability, existence of positive solution to a system of linear algebraic inequalities and the fact that a matrix constructed from the coefficients is Hurwitz for system of ordinary differential equations with Metzler matrix (see Definition 2.2 and Proposition 2.2). The presented approach allows to stabilize unstable state-delay systems by feedback with large input delays. The corresponding result is proved under assumption about nonoscillation of the "diagonal" scalar delay equations in Theorem 3.5. A principal possibility to achieve stabilization of system (3.22) (see below) by the feedback control (3.23), where the delays $\tau_{ij}(t)$ are greater than the state delays $\theta_{ij}(t)$ of (3.22), is formulated in Corollaries 3.1 and 3.2. The stability results are formulated in terms of inequalities on the delays and on the coefficients.

The present paper is organized as follows. In Section 2, we discuss positivity-based methods in the stability analysis. In Section 3, we formulate our main results. In Section 4, the proofs of the main results are given.

Notations: Throughout the paper e denotes the Euler number. L_∞ is the space of essentially bounded measurable functions $y : [0,+\infty) \to \mathbb{R}$. For $y \in L_\infty$ denote $y^* = \operatorname{esssup}_{t\geq 0} y(t)$, $y_* = \operatorname{essinf}_{t\geq 0} y(t)$ and for $y^k \in L_\infty (k = 1,\dots,m)$ − $y^+(t) = \max_{k=1,\dots,m}\{y^k(t)\}$, $y^-(t) = \min_{k=1,\dots,m}\{y^k(t)\}$.

2. Preliminaries on positivity and stability of time-delay systems

Consider the non-homogeneous system

$$x'(t) - \sum_{k=1}^{m} A_k(t)x(t-\theta_k(t)) = f(t), \quad t \in [0,+\infty), \tag{2.1}$$

$$x(\xi) = 0, \quad \xi < 0, \tag{2.2}$$

where $A_k(t) = \{a_{ij}^k(t)\}_{i,j=1,\dots,n}$ are $n \times n$ matrices with entries $a_{ij}^k \in L_\infty$, $\theta_k \in L_\infty$ for $k = 1,\dots,m$, $f(t) = col\{f_1(t),\dots,f_n(t)\}$, $f_i \in L_\infty$, for $i = 1,\dots,n$. The components $x_i : [0,+\infty) \to \mathbb{R}$ of the vector $x = col\{x_1,\dots,x_n\}$, are assumed to be absolutely continuous and their derivatives $x_i' \in L_\infty$. A vector-function x is a solution of (2.1) if it satisfies system (2.1) for almost all $t \in [0,+\infty)$.

It was explained in [24] that without loss of generality, the zero initial condition (2.2) can be considered instead of (1.2). The homogeneous system

$$x'(t) - \sum_{k=1}^{m} A_k(t)x(t-\theta_k(t)) = 0, \quad t \in [0,+\infty), \tag{2.3}$$

with initial function defined by (2.2), has n-dimensional space of solutions [24] and this fact is the basis of solutions' representations which will be used below.

Let us define the Cauchy matrix $C(t,s) = \{C_{ij}(t,s)\}_{i,j=1,\dots,n}$ as follows [24]. For every fixed $s \geq 0$, as a function of the variable t, it satisfies the matrix equation

$$C_t'(t,s) = \sum_{k=1}^{m} A_k(t)C(t-\theta_k(t),s), \quad t \in [s,+\infty), \tag{2.4}$$

where

$$C(\xi,s) = 0, \quad \text{for } \xi < s, \tag{2.5}$$

and

$$C(s,s) = I. \tag{2.6}$$

I is the unit matrix. The general solution of system (2.1), (2.2) can be represented in the form [24]

$$x(t) = \int_0^t C(t,s)f(s)ds + C(t,0)x(0). \tag{2.7}$$

Definition 2.1. The Cauchy matrix $C(t,s)$ is said to satisfy the exponential estimate if there exist positive numbers N and α such that

$$\left|C_{ij}(t,s)\right| \leq N\exp\{-\alpha(t-s)\}, \quad i,j = 1,\dots,n, \quad 0 \leq s \leq t < +\infty. \tag{2.8}$$

In this case we say that (2.3) is exponentially stable.

Our main results will be based on the following extension of the classical Bohl–Perron theorem:

Proposition 2.1 ([4]). *In the case of bounded delays $\theta_k(t)$ and coefficients in the matrices $A_k(t)$ $(k = 1,\dots,m)$, the fact that for every bounded right-hand side $f(t) = col\{f_1(t),\dots,f_n(t)\}$, the solution $x(t) = col\{x_1(t),\dots,x_n(t)\}$ of system (2.1) is bounded on the semiaxis $[0,+\infty)$ is equivalent to the exponential estimate (2.8) of the Cauchy matrix $C(t,s)$.*

T. Wazewski [5] proved that for system of ordinary differential equations $x'(t) = A(t)x(t)$ the nonnegativity of all off-diagonal elements of $A(t)$

$$a_{ij}(t) \geq 0 \quad \text{for } i \neq j,\ i,j = 1,\dots,n,\ t \in [0,+\infty), \tag{2.9}$$

is necessary and sufficient for the nonnegativity of all entries of the Cauchy matrix $C(t,s) = \{C_{ij}(t,s)\}_{i,j=1,\dots,n}$ of the system.

Definition 2.2. The matrix A is Metzler if all its off-diagonal elements are nonnegative for $t \geq 0$, i.e. (2.9) is fulfilled.

The fact that all matrices $A_k(t)$ are Metzler together with the smallness of diagonal delays (see condition (2.12)) implies $C_{ij}(t,s) \geq 0$ for $0 \leq s \leq t < +\infty$, $i,j = 1,\dots,n$ [25,26]. In Theorems 3.1 and 3.2 of the present paper, we propose new assumptions on the diagonal delay differential equations (actually, nonoscillation of their solutions), which together with the condition that the matrices $A_k(t)$ are Metzler, imply the nonnegativity of $C(t,s)$.